MATH162 - Summer 2007/2008
Outline Solutions to Tutorial Sheet - Week 6

1. (a) $1, \frac{5}{4}, \frac{7}{5}, \frac{3}{2}, \frac{11}{7}$

$$\lim_{n\to\infty} \frac{2n+1}{n+2} = 2$$

Thus, the sequence is convergent.

(b) $1, \frac{4}{3}, \frac{9}{5}, \frac{16}{7}, \frac{25}{9}$

$$\lim_{n\to\infty} \frac{n^2}{2n-1} = \lim_{n\to\infty} \frac{n}{2-\frac{1}{n}}$$

This limit does not exist, thus the sequence is divergent.

(c) $0, -\ln 2, -\ln 3, -\ln 4, -\ln 5$

$$\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = \lim_{n\to\infty}(-\ln n)$$

This limit does not exist, thus the sequence is divergent.

(d) $-1, \frac{16}{9}, -\frac{54}{28}, \frac{128}{65}, -\frac{250}{126}$

$$\lim_{n\to\infty} \frac{(-1)^n 2n^3}{n^3+1} = \lim_{n\to\infty}(-1)^n \times 2$$

This limit does not exist, thus the sequence is divergent.

(e) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}$

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n^2} = 0$$

Thus, the sequence is convergent.

(f) $\cos 1, \frac{\cos 2}{2}, \frac{\cos 3}{3}, \frac{\cos 4}{4}, \frac{\cos 5}{5}$

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0 \text{ (by ‘Squeeze Law’)}$$

Thus, the sequence is convergent.

(g) $\sinh n - \cosh n = \frac{e^n - e^{-n}}{2} - \frac{e^n + e^{-n}}{2} = -e^{-n}$.

The first 5 terms are $-e^{-1}$, $-e^{-2}$, $-e^{-3}$, $-e^{-4}$, $-e^{-5}$.

Now, $\lim_{n\to\infty}(\sinh n - \cosh n) = -\lim_{n\to\infty} \frac{1}{e^n} = 0$, and the sequence is convergent.

2. (a) The series $\sum_{k=1}^{\infty} \frac{1}{5^k}$ is a geometric series with $r = \frac{1}{5} < 1$.

Therefore, the series converges to $S = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$.

(b) The series $\sum_{k=1}^{\infty} \left(\frac{-3}{2}\right)^{k+1}$ is a geometric series with $|r| = \frac{3}{2} \geq 1$.

Therefore, the series diverges.

(c) We must form the sequence of partial sums. Note that $\frac{1}{9k^2+3k-2} = \frac{1}{3}\left(\frac{1}{3k-1} - \frac{1}{3k+2}\right)$.

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{3}\left(\frac{1}{3k-1} - \frac{1}{3k+2}\right) \\ &= \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) + \ldots + \left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)\right] \\ &= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{3n+2}\right]. \end{aligned}$$

Now, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{6}$.

Therefore, the series $\sum_{k=1}^{\infty} \frac{1}{9k^2+3k-2}$ converges to $\frac{1}{6}$.

3. (a) Take $u_n = \dfrac{1}{n^3+2n}$. Consider $c_n = \dfrac{1}{n^3}$, where $\sum_{n=1}^{\infty} c_n$ converges (p-series, $p = 3 > 1$).

$$\begin{aligned}\lim_{n\to\infty}\frac{u_n}{c_n} &= \lim_{n\to\infty}\frac{1}{n^3+2n}\times\frac{n^3}{1}\\ &= \lim_{n\to\infty}\frac{n^3}{n^3+2n}\\ &= \lim_{n\to\infty}\frac{1}{1+\frac{2}{n^2}} = 1, \text{ which is finite.}\end{aligned}$$

Therefore, by the Comparison Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{1}{n^3+2n}$ converges.

(b) Take $u_n = \dfrac{9}{\sqrt{n}+1}$. Consider $d_n = \dfrac{1}{\sqrt{n}}$, where $\sum_{n=1}^{\infty} d_n$ diverges (p=series, $p = \frac{1}{2} \le 1$).

$$\begin{aligned}\lim_{n\to\infty}\frac{u_n}{d_n} &= \lim_{n\to\infty}\frac{9}{\sqrt{n}+1}\times\frac{\sqrt{n}}{1}\\ &= \lim_{n\to\infty}\frac{9\sqrt{n}}{\sqrt{n}+1}\\ &= \lim_{n\to\infty}\frac{9}{1+\frac{1}{\sqrt{n}}} = 9 > 0.\end{aligned}$$

Therefore, by the Comparison Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{9}{\sqrt{n}+1}$ diverges.

(c) First, $(\cos n)^2 \le 1$. Thus, $\dfrac{(\cos n)^2}{n^2} \le \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$. Also $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ is a known convergent series.

Therefore, $\sum_{n=1}^{\infty}\left(\dfrac{\cos n}{n}\right)^2$ converges by the Comparison Test.

(d) For all n, $n + 3^n \ge 3^n$ and so $\dfrac{1}{n+3^n} \le \dfrac{1}{3^n} \implies \dfrac{2^n}{n+3^n} \le \dfrac{2^n}{3^n}$

Now, $\sum_{n=1}^{\infty}\left(\dfrac{2}{3}\right)^n$ is a known convergent series (geometric with $r = \frac{2}{3} < 1$).

Therefore, $\sum_{n=1}^{\infty}\dfrac{2^n}{n+3^n}$ converges by the Comparison Test.

4. (a) Consider $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\dfrac{n}{\ln n}$ $\left(=\frac{\infty}{\infty}\right)$. Now, $\lim_{x\to\infty}\dfrac{x}{\ln x} = \lim_{x\to\infty}\dfrac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty \ne 0$

Therefore, $\lim_{n\to\infty}\dfrac{n}{\ln n} \ne 0$ and so $\sum_{n=1}^{\infty}\dfrac{n}{\ln n}$ diverges by the nth term test.

(b) $u_n = \dfrac{n^2}{5^n}$, $u_{n+1} = \dfrac{(n+1)^2}{5^{n+1}}$

$$\begin{aligned}\rho &= \lim_{n\to\infty}\frac{u_{n+1}}{u_n}\\ &= \lim_{n\to\infty}\frac{(n+1)^2}{5^{n+1}}\times\frac{5^n}{n^2}\\ &= \lim_{n\to\infty}\frac{(n+1)^2}{5n^2}\\ &= \lim_{n\to\infty}\frac{n^2+2n+1}{5n^2} = \frac{1}{5}\end{aligned}$$

So by d'Alembert's ratio test, the series $\sum_{n=1}^{\infty}\dfrac{n^2}{5^n}$ is convergent.

(c) $u_n = \dfrac{n!}{n^3}$, $u_{n+1} = \dfrac{(n+1)!}{(n+1)^3}$

$$\begin{aligned}\rho &= \lim_{n\to\infty}\frac{u_{n+1}}{u_n}\\ &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^3}\times\frac{n^3}{n!}\\ &= \lim_{n\to\infty}\frac{(n+1)n^3}{(n+1)^3}\\ &= \lim_{n\to\infty}\frac{n^4+n^3}{n^3+3n^2+3n+1} \quad \text{DNE}\end{aligned}$$

So by d'Alembert's ratio test, the series $\sum_{n=1}^{\infty}\dfrac{n!}{n^3}$ is divergent.

5. (a) $u_n = \dfrac{1}{n - \ln n}$ Let $d_n = \dfrac{1}{n}$. Now, $\displaystyle\lim_{n\to\infty} \frac{u_n}{d_n} = \lim_{n\to\infty} \frac{n}{n - \ln n} = \lim_{n\to\infty} \frac{1}{1 - \frac{\ln n}{n}} = 1 > 0$

Hence, by the comparison ratio test,the series is divergent.

(b) Applying the n^{th} term test, we have:

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \cos n\pi \\ &= \lim_{n\to\infty} (-1)^n \neq 0.\end{aligned}$$

Therefore, by the n^{th} term test, the series is divergent.

(c) $u_n = \dfrac{(n!)^2}{(2n)!}$, $u_{n+1} = \dfrac{((n+1)!)^2}{(2n+2)!}$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{u_{n+1}}{u_n} \\ &= \lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(2n+2)!} \times \frac{(2n)!}{n!n!} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \\ &= \frac{1}{4} < 1\end{aligned}$$

So by d'Alembert's ratio test, the series is convergent.

6. (a) Check for absolute convergence: $|u_n| = \dfrac{1}{2n-1}$, $d_n = \dfrac{1}{n}$.

Then, $\displaystyle\lim_{n\to\infty} \frac{|u_n|}{d_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$

Therefore, by the Comparison Ratio Test, the series $\displaystyle\sum_{n=1}^{\infty} |u_n|$ diverges.

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ is not absolutely convergent.

Check the conditions for the Alternating Series Test:

(i) $u_n = \frac{(-1)^{n+1}}{2n-1}$ so alternating.

(ii) $\displaystyle\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0$

(iii) $u_n = (-1)^{n+1} f(n)$ where $f(x) = \dfrac{1}{2x-1}$.

$$f'(x) = \frac{(-1)(2)}{(2x-1)^2} = \frac{-2}{(2x-1)^2} < 0 \text{ for } x \geq 1 \implies |u_{n+1}| \leq |u_n|.$$

Therefore, the series satisfies the Alternating Series Test.

Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ is conditionally convergent.

(b) $u_n = (-1)^n \dfrac{3n+2}{7n-3}$. So, consider the limit of $|u_n|$:

$$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{3n+2}{7n-3} = \frac{3}{7}.$$

Therefore, $\displaystyle\lim_{n\to\infty} u_n$ does not exist. Thus, by the nth term test, $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3n+2}{7n-3}$ is divergent.

(c) The series is alternating, so we can try the ratio test for absolute convergence:

$|u_n| = \dfrac{4^n}{n^2}$ and $|u_{n+1}| = \dfrac{4^{n+1}}{(n+1)^2}$

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| \\ &= \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)^2} \times \frac{n^2}{4^n} \\ &= \lim_{n\to\infty} 4 \times \frac{n^2}{n^2+2n+1} \\ &= 4 \quad > 1\end{aligned}$$

Hence the series is divergent by the Ratio test.