MATH162 - Summer 2007/2008
Outline Solutions to Tutorial Sheet - Week 4

1. (a) (i) 1st order, 1st degree.
(ii) Dependent variable: x;
Independent variable: t.
(iii) Linear with non-constant coefficients.

(b) (i) 4th order, 1st degree.
(ii) Dependent variable: y;
Independent variable: Not specified.
(iii) Non-linear.

(c) (i) 2nd order, 1st degree.
(ii) Dependent variable: u; Independent variable: t.
(iii) Non-linear.

2. (a) $y = ce^{4x} \implies y' = 4ce^{4x}$.

$$\begin{aligned} \text{Therefore, LHS } &= 4ce^{4x} \\ &= 4y \\ &= \text{RHS.} \end{aligned}$$

(b) $y = c_1 \cos x + c_2 \sin x$

$$\begin{aligned} &\implies y' = -c_1 \sin x + c_2 \cos x \\ &\implies y'' = -c_1 \cos x - c_2 \sin x. \end{aligned}$$

$$\begin{aligned} \text{Thus, LHS } &= -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \sin x \\ &= 0 \\ &= \text{RHS.} \end{aligned}$$

(c) $y = 5\tan 5x \implies y' = 25\sec^2 5x$.

$$\begin{aligned} \text{Now, RHS } &= 25 + 25\tan^2 5x \\ &= 25\sec^2 5x \\ &= y' = \text{LHS.} \end{aligned}$$

3. $$y' = \frac{1}{3}\left(\frac{3}{2}x^2 + k\right)^{-2/3} \times 3x \implies y' = \frac{x}{\left(\frac{3}{2}x^2 + k\right)^{2/3}} = \frac{x}{\left(\left(\frac{3}{2}x^2 + k\right)^{1/3}\right)^2}.$$

Therefore, $y' = \dfrac{x}{y^2}$ is the least order DE.

4. $$\frac{dy}{dx} = \frac{2x^2y + y}{x} \implies \frac{dy}{dx} = y\left(\frac{2x^2 + 1}{x}\right)$$ Variables separable. Therefore, we have

$$\frac{dy}{y} = \frac{2x^2 + 1}{x}dx \implies \frac{dy}{y} = \left(2x + \frac{1}{x}\right)dx.$$

Integrating, $\ln y = x^2 + \ln x + c \implies y = Axe^{x^2}$.

When $x = 1$ and $y = 2$, we have: $2 = A.1.e \implies A = 2e^{-1}$.

Therefore, $y = \dfrac{2}{e}xe^{x^2}$.

5. (a) $xy' + 3y = x^2$ is linear in y and we can write $y' + \dfrac{3}{x}y = x$, with $p(x) = \dfrac{3}{x}$ and $q(x) = x$.

We calculate the integrating factor $R(x) = \exp\left(\displaystyle\int \frac{3}{x}dx\right) = e^{3\ln x} = x^3$.

Now we solve $\dfrac{d}{dx}(Ry) = Rq(x)$, i.e., $\dfrac{d}{dx}\left(x^3y\right) = x^3.x = x^4$.

Integrating, we get $x^3y = \dfrac{x^5}{5} + c$.

(b) $$y' = \frac{xy}{x^2 - y^2} = \frac{y/x}{1 - (y/x)^2} \quad \text{(homogeneous).}$$

Let $v = \dfrac{y}{x}$. Then $y = vx$ and $y' = v'x + v$, and the DE becomes

$$v'x + v = \frac{v}{1 - v^2} \implies v'x = \frac{v - v + v^3}{1 - v^2} \implies dv\,\frac{(1 - v^2)}{v^3} = \frac{dx}{x}.$$

Integrating: $$\int \left(v^{-3} - v^{-1}\right) dv = \int \frac{dx}{x} \implies -\frac{v^{-2}}{2} - \ln|v| = \ln|x| + c$$

$$\implies -\frac{x^2}{2y^2} - \ln\left|\frac{y}{x}\right| = \ln|x| + c, \quad \text{or} \quad e^{-\frac{x^2}{2y^2}} = Ay.$$

(c) $y' - y\tan x = 1$ is linear with $p(x) = -\tan x$ and $q(x) = 1$.

We need to find the integrating factor: $R(x) = \exp\left(\int -\dfrac{\sin x}{\cos x}dx\right) = \cos x.$

Thus we have $\dfrac{d}{dx}(y\cos x) = \cos x$

Integrating, we get $y\cos x = \sin x + c$

6. (a) Linear and Separable (b) Separable (c) Linear

(d) Linear and Separable (e) Homogeneous

7. (a) Auxiliary Equation: $m^2 - 4 = 0 \implies m = \pm 2$

Therefore, $y = y_c = Ae^{2x} + Be^{-2x}$.

(b) Auxiliary Equation: $m^2 + m + 1 = 0 \implies m = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

Therefore, $y = e^{-\frac{1}{2}x}\left(A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right)$.

(c) Auxiliary Equation: $m^2 - 2m + 1 = 0 \implies m = 1, 1$

Therefore, $y = (A + Bx)e^x$.

8. For complementary function y_c: $m^2 + 4m + 9 = 0 \implies m = -2 \pm \sqrt{5}i$

Therefore, $y_c = e^{-2x}\left(A\cos\sqrt{5}x + B\sin\sqrt{5}x\right)$

For particular integral y_p:

$$\begin{aligned} y_p &= \frac{1}{\mathrm{D}^2 + 4\mathrm{D} + 9}\left(x^2 + 3x\right) \qquad \ldots \text{ Type 1} \\ &= \frac{1}{9\left(1 + \frac{4\mathrm{D}}{9} + \frac{\mathrm{D}^2}{9}\right)}\left(x^2 + 3x\right) \end{aligned}$$

This can be done by binomial expansion or long division.

Using Long Division:

$$\begin{array}{r|l} & 1 \; - \; \frac{4\mathrm{D}}{9} \; + \; \frac{7}{81}\mathrm{D}^2 + \ldots \\ \hline 1 + \frac{4\mathrm{D}}{9} + \frac{\mathrm{D}^2}{9} & 1 \\ & 1 \; + \; \frac{4\mathrm{D}}{9} \; + \; \frac{\mathrm{D}^2}{9} \\ \cline{2-2} & \quad - \frac{4\mathrm{D}}{9} - \frac{\mathrm{D}^2}{9} \\ & \quad - \frac{4\mathrm{D}}{9} - \frac{16}{81}\mathrm{D}^2 \\ \cline{2-2} & \qquad\qquad \frac{7}{81}\mathrm{D}^2 \end{array}$$

Therefore, $y_p = \dfrac{1}{9}\left(1 - \dfrac{4\mathrm{D}}{9} + \dfrac{7}{81}\mathrm{D}^2\right)\left(x^2 + 3x\right) = \dfrac{1}{9}\left(x^2 + \dfrac{19}{9}x - \dfrac{94}{81}\right)$

Therefore, $y = y_c + y_p = e^{-2x}\left(A\cos\sqrt{5}x + B\sin\sqrt{5}x\right) + \dfrac{1}{9}\left(x^2 + \dfrac{19}{9}x - \dfrac{94}{81}\right)$

9. (a) Auxiliary equation: $m^2 + 3m + 2 = 0 \implies m = -2, -1$

Hence $y_c = Ae^{-2x} + Be^{-x}$.

$$\begin{aligned} y_p &= \frac{1}{\mathrm{D}^2 + 3\mathrm{D} + 2}.e^{4x} \qquad \ldots \text{ Type 2, } a = 4 \\ &= \frac{e^{4x}}{4^2 + 3\times 4 + 2} = \frac{e^{4x}}{30} \end{aligned}$$

Therefore, $y = y_c + y_p = Ae^{-2x} + Be^{-x} + \dfrac{e^{4x}}{30}$.

(b) Auxiliary equation: $m^2 + 4m + 5 = 0 \implies m = -2 \pm i$

Hence $y_c = e^{-2x}(A\sin x + B\cos x)$.

Continued over...

$$y_p = \frac{1}{D^2+4D+5}\cdot 8\cos x \qquad \ldots \text{ Type 4, } a=1,\ D^2 \to -(1^2) = -1$$

$$= \frac{1}{-1+4D+5}\cdot 8\cos x$$

$$= \frac{8}{4}\cdot\frac{1}{D+1}\cdot\frac{D-1}{D-1}\cdot\cos x$$

So $y_p = 2\,\dfrac{D-1}{D^2-1}\cdot\cos x$

$$= 2\,\frac{D-1}{-2}\cdot\cos x$$

$$= -1(-\sin x - \cos x) = \sin x + \cos x$$

Therefore, $y = y_c + y_p = e^{-2x}(A\sin x + B\cos x) + \sin x + \cos x$.

(c) Auxiliary equation:

$$m^2 + 4m = 0 \implies m = 0, -4$$

Hence $y_c = Ae^{0x} + Be^{-4x}$

$$= A + Be^{-4x}.$$

$$y_p = \frac{1}{D^2+4D}\cdot x^3 \qquad \ldots \text{ Type 1}$$

$$= \frac{1}{D}\frac{1}{D+4}\cdot x^3$$

$$= \frac{1}{D}\frac{1}{4}\frac{1}{1+\frac{D}{4}}\cdot x^3$$

$$= \frac{1}{4}\frac{1}{D}\left[1 - \frac{D}{4} + \frac{D^2}{16} - \frac{D^3}{64}\right]\cdot x^3$$

$$= \frac{1}{4}\frac{1}{D}\cdot\left[x^3 - \frac{3x^2}{4} + \frac{3x}{8} - \frac{3}{32}\right]$$

$$= \frac{1}{4}\left[\frac{x^4}{4} - \frac{x^3}{4} + \frac{3x^2}{16} - \frac{3x}{32}\right]$$

Therefore, $y = y_c + y_p = A + Be^{-4x} + \dfrac{1}{4}\left[\dfrac{x^4}{4} - \dfrac{x^3}{4} + \dfrac{3x^2}{16} - \dfrac{3x}{32}\right]$.

10. Solving as 2nd order y-absent:

Let $\rho = \dfrac{dy}{dx} \implies \dfrac{d\rho}{dx} = \dfrac{d^2y}{dx^2}$.

Therefore, $\dfrac{d\rho}{dx} - \rho = e^{2x}$ which is linear.

Find the integrating factor $R = e^{-x}$.

Therefore, we have $\dfrac{d}{dx}(\rho e^{-x}) = e^{x} \implies \rho e^{-x} = e^{x} + A$.

Hence, $\dfrac{dy}{dx} = e^{2x} + Ae^{x} \implies y = \dfrac{e^{2x}}{2} + Ae^{x} + B$.

Solving as 2nd order linear with constant coefficients:

For complementary function y_c: $m^2 - m = 0 \implies m(m-1) = 0 \implies m = 0, 1$

Therefore, $y_c = Ae^{0x} + Be^{x} = A + Be^{x}$

For particular integral y_p:

$$y_p = \frac{1}{D^2 - D}e^{2x} \qquad \ldots \text{ Type 2}$$

$$= \frac{e^{2x}}{2^2 - 2}$$

$$= \frac{e^{2x}}{2}.$$

Therefore, $y = y_c + y_p = A + Be^{x} + \dfrac{e^{2x}}{2}$.

The solutions for the different methods are the same!

11. 2nd order, x-absent. Let $\rho = \dfrac{dy}{dx} \implies \rho\dfrac{d\rho}{dy} = \dfrac{d^2y}{dx^2}$.

Therefore, $2y\rho\dfrac{d\rho}{dy} = 1 + \rho^2 \implies \dfrac{d\rho}{dy} = \dfrac{1+\rho^2}{2y\rho}$ (separable) $\implies \dfrac{2\rho}{1+\rho^2}\,d\rho = \dfrac{1}{y}\,dy$

Integrate $\ln\left(1+\rho^2\right) = \ln y + c \quad (c = \ln A)$

or $\rho = \pm\sqrt{Ay - 1}$

Hence, $\dfrac{dy}{dx} = \pm\sqrt{Ay-1}$ (separable) $\implies \dfrac{dy}{\sqrt{Ay-1}} = \pm dx$

Integrate $2\sqrt{Ay-1} = \pm Ax + B$