MATH162 - Summer 2007/2008
Outline Solutions to Tutorial Sheet - Week 2

1. (a) (i) $f(x) = 1$ implies $\int_0^1 1\,dx = 1 = w_1 \cdot 1 + w_2 \cdot 1$

(ii) $f(x) = x$ implies $\int_0^1 x\,dx = \frac{1}{2} = w_1 \cdot 0 + w_2 \cdot 1.$

(b) Clearly, $w_2 = \frac{1}{2}$, and so, $w_1 = \frac{1}{2}$. Hence $I = \int_0^1 f(x)\,dx \approx \frac{1}{2}f(0) + \frac{1}{2}f(1).$

(c)
$$\begin{aligned}\int_0^1 1 + 2x\,dx &\approx \frac{1}{2} \times (1 + 2 \times 0) + \frac{1}{2} \times (1 + 2 \times 1)\\ &= \frac{1}{2} + \frac{3}{2} = 2.\end{aligned}$$

(d) The exact solution is $\int_0^1 1 + 2x\,dx = \left[x + x^2\right]_0^1 = 2$. Therefore the absolute error is 0.

(e)
$$\begin{aligned}\int_0^1 x^2\,dx &\approx \frac{1}{2}f(0) + \frac{1}{2}f(1)\\ &= \frac{1}{2} \times 0 + \frac{1}{2} \times 1\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\int_0^1 x^2\,dx &= \left[\frac{x^3}{3}\right]_0^1\\ &= \frac{1}{3}\end{aligned}$$

Therefore the absolute error is $\left|\frac{1}{3} - \frac{1}{2}\right| = \frac{1}{6}.$

2. For $f(x) = 1$: $\int_0^1 1dx = \omega_1 \times 1 + \omega_2 \times 1 \implies 1 = \omega_1 + \omega_2$ ...(1)

For $f(x) = x$: $\int_0^1 xdx = \omega_1 \times 0 + \omega_2 \times x_1 \implies \frac{1}{2} = \omega_2 x_1$ ...(2)

For $f(x) = x^2$: $\int_0^1 x^2dx = \omega_1 \times 0 + \omega_2 \times x_1^2 \implies \frac{1}{3} = \omega_2 x_1^2$ ...(3)

Solving the system: Dividing (3) by (2): $x_1 = \frac{2}{3}$

Substituting into (2): $\omega_2 = \frac{3}{4}$

Substituting into (1): $\omega_1 = \frac{1}{4}.$

So the quadrature scheme is $\int_0^1 f(x)dx = \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right).$

3. (a)
$$\begin{aligned}\int_0^1 \frac{4dx}{1+x^2} &\approx \frac{h}{2}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big)\\ &= \frac{0.1}{2}\left(\frac{4}{1.00} + 2 \times \frac{4}{1.01} + 2 \times \frac{4}{1.04} + \cdots + 2 \times \frac{4}{1.81} + \frac{4}{2}\right)\\ &\approx 3.139925990\end{aligned}$$

The exact value of the integral is $4 \arctan 1 = \pi \approx 3.141592653$,
so the error is approximately $\left|3.139925990 - 3.141592653\right| = 0.001666663.$

(b)
$$\begin{aligned}\int_0^1 \frac{4dx}{1+x^2} &\approx \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right)\\ &= \frac{2}{1 + \left(\frac{1}{4}\right)^2} + \frac{2}{1 + \left(\frac{3}{4}\right)^2}\\ &= \frac{1344}{425} \approx 3.162352941.\end{aligned}$$

The error is approximately
$\left|3.162352941 - 3.141592653\right| = 0.020760288.$

4. There are $n = 8$ intervals from 3 to 5. Thus, the step size is $h = \dfrac{5-3}{8} = 0.25$.
Simpson's Rule gives

$$\begin{aligned}
\int_3^5 y\ dx &\approx S_8 \\
&= \frac{h}{3}\Big[f(3) + 4f(3.25) + 2f(3.5) + 4f(3.75) + 2f(4.0) + 4f(4.25) + 2f(4.5) + 4f(4.75) + f(5)\Big] \\
&= \frac{0.25}{3}\Big[6.7 + 4(7.4) + 2(8.2) + 4(9.2) + 2(10.4) + 4(11.6) + 2(12.5) + 4(13.3) + 14.0\Big] \\
&\approx 20.7.
\end{aligned}$$

5. (a)
$$\begin{aligned}
\int xe^{x^2}\ dx &= \int \frac{1}{2}e^u du \qquad \text{by letting } u = x^2 \quad \Longrightarrow \quad \tfrac{du}{2} = x\,dx \\
&= \frac{1}{2}e^u + c \\
&= \frac{1}{2}e^{x^2} + c.
\end{aligned}$$

(b)
$$\begin{aligned}
\int \sinh^3 x \cosh x\, dx &= \int u^3 du \qquad \text{by letting } u = \sinh x \quad \Longrightarrow \quad du = \cosh x\, dx \\
&= \frac{u^4}{4} + c \\
&= \frac{\sinh^4 x}{4} + c
\end{aligned}$$

(c)
$$\begin{aligned}
\int \cosh^2 x \sinh x\, dx &= \int u^2 du \qquad \text{by letting } u = \cosh x \quad \Longrightarrow \quad du = \sinh x\, dx \\
&= \frac{u^3}{3} + c \\
&= \frac{\cosh^3 x}{3} + c
\end{aligned}$$

6. (a)
$$\begin{aligned}
&\int_0^1 (x^2-1)e^{x^3-3x} dx \\
&= \frac{1}{3}\int_0^{-2} e^u\ du \\
&= \frac{1}{3}\Big[e^u\Big]_0^{-2} \\
&= \frac{1}{3}(e^{-2} - 1).
\end{aligned}$$

Let $u = x^3 - 3x$, then $\dfrac{du}{3} = (x^2-1)dx$ and when $x = 0$, then $u = 0$ and when $x = 1$, $u = -2$.

(b)
$$\begin{aligned}
\int \frac{dx}{\sqrt{1-16x^2}} &= \int \frac{dx}{\sqrt{1-(4x)^2}} \\
&= \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{4}\sin^{-1} u + c \\
&= \frac{1}{4}\sin^{-1}(4x) + c.
\end{aligned}$$

Let $u = 4x$ then $\dfrac{du}{4} = dx$.

(c) $\displaystyle\int \sin^2 x\ dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + c$ [33] Table of Integrals.

(d) $\displaystyle\int_{\pi/6}^{\pi/4} \tan\theta \sec^2\theta d\theta$

Let $u = \tan\theta$, then $du = \sec^2\theta\ d\theta$.

When $\theta = \frac{\pi}{4}$, $u = 1$, and when $\theta = \frac{\pi}{6}$, $u = \frac{1}{\sqrt{3}}$.

$$= \int_{1/\sqrt{3}}^{1} u\ du$$

$$= \left[\frac{u^2}{2}\right]_{1/\sqrt{3}}^{1}$$

$$= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{1}{3}$$

(e) $\displaystyle\int_{\pi^2/16}^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Let $u = \sqrt{x}$, then $2du = \frac{dx}{\sqrt{x}}$.

When $x = \frac{\pi^2}{4}$, then $u = \frac{\pi}{2}$; and when $x = \frac{\pi^2}{16}$, $u = \frac{\pi}{4}$.

$$= 2\int_{\pi/4}^{\pi/2} \sin u du$$

$$= 2\Big[-\cos u\Big]_{\pi/4}^{\pi/2}$$

$$= 2(-\cos\pi/2 + \cos\pi/4)$$

$$= \sqrt{2}$$

7. (a) $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$

Let $x = \sec\theta$, then $x^2 - 1 = \tan^2\theta$ and $dx = \sec\theta\tan\theta\ d\theta$.

Now, $x = \frac{1}{\cos\theta}$ and so $\cos\theta = \frac{1}{x}$.

Thus, when $x = 2$, $\theta = \frac{\pi}{3}$ and when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\tan\theta}$$

$$= \int_{\pi/4}^{\pi/3} \cos\theta d\theta$$

$$= \Big[\sin\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$$

(b) $\displaystyle\int \frac{dx}{(4+x^2)^2}$

Let $x = 2\tan\theta$, then $4 + x^2 = 4\sec^2\theta$ and $dx = 2\sec^2\theta\ d\theta$.

$$= \int \frac{2\sec^2\theta d\theta}{(4\sec^2\theta)^2}$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta} d\theta$$

$$= \frac{1}{8}\int \cos^2\theta d\theta$$

$$= \frac{1}{16}[\sin\theta\cos\theta + \theta] + c \qquad [34]$$

$$= \frac{1}{16}\left[\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}\right] + c \text{ (from triangle)}$$