MATH162 - Summer 2007/2008
Outline Solutions to Tutorial Sheet - Week 1

1. (a)

$$\begin{aligned}\frac{(1+i)(1-2i)}{(2+i)(4-3i)} &= \frac{1-i+2}{8-2i+3}\\ &= \frac{3-i}{11-2i}\times\frac{11+2i}{11+2i}\\ &= \frac{35-5i}{125} = \frac{7}{25}-\frac{1}{25}i\end{aligned}$$

(b)

$$\begin{aligned}\left(\frac{i}{3-i}\right)\times\left(\frac{1}{2+3i}\right) &= \frac{i}{6+7i+3}\\ &= \frac{i}{9+7i}\times\frac{9-7i}{9-7i}\\ &= \frac{7+9i}{81+49}\\ &= \frac{7}{130}+\frac{9}{130}i\end{aligned}$$

2. If $z = x+iy$, then $\overline{z} = x-iy$.

(a)

$$\begin{aligned}\frac{1}{2}(z+\overline{z}) &= \frac{1}{2}(x+iy+x-iy)\\ &= x\\ &= \mathrm{R}(z)\end{aligned}$$

(b)

$$\begin{aligned}z = -\overline{z} \quad &\Longrightarrow \quad x+iy = -(x-iy)\\ &\qquad\qquad\;\; = -x+iy\\ &\Longrightarrow \quad x = -x\\ &\Longrightarrow \quad 2x = 0 \quad \Longrightarrow \quad x = 0.\end{aligned}$$

Therefore, $z = iy$. That is, z is purely imaginary.

3. Using the quadratic formula,

$$\begin{aligned}z &= -1\pm\frac{1}{2}\sqrt{4-8}\\ &= -1\pm\sqrt{-1}\\ &= -1\pm i.\end{aligned}$$

4. (a)

$$\begin{aligned}e^z &= e^2e^{-5\pi i/4}\\ &= e^2\left(\cos\tfrac{5\pi}{4}-i\sin\tfrac{5\pi}{4}\right)\\ &= \frac{e^2}{\sqrt{2}}(-1+i)\end{aligned}$$

(b)

$$\begin{aligned}e^z &= e^1e^{\pi/3}\\ &= e\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right)\\ &= \frac{e}{2}(1+i\sqrt{3}).\end{aligned}$$

5. (a)

$$\begin{aligned}r &= \sqrt{(-1)^2+(\sqrt{3})^2}\\ &= 2\end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1}\left(-\sqrt{3}\right)\\ &= \frac{2\pi}{3}\end{aligned}$$

[Check quadrant: 2nd]

Therefore, $z = 2\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$.

$$\begin{aligned}\text{Now, } z^{10} &= \left[2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right]^{10}\\ &= 2^{10}\left(\cos\frac{10\times2\pi}{3}+i\sin\frac{10\times2\pi}{3}\right)\\ &= 2^{10}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\ &= 2^9(-1+i\sqrt{3})\end{aligned}$$

(b)

$$\begin{aligned}|z| &= \sqrt{2^2+(-2)^2}\\ &= 2\sqrt{2}\end{aligned}$$

$$\begin{aligned}\arg(z) &= \tan^{-1}\left(\frac{-2}{2}\right)\\ &= \frac{7\pi}{4}\end{aligned}$$

[Check quadrant: 4th]

Therefore, $z = 2^{3/2}\left(\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}\right)$.

$$\begin{aligned}\text{Now, } z^{5} &= \left[2^{3/2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]^{5}\\ &= 2^{15/2}\left(\cos\frac{35\pi}{4}+i\sin\frac{35\pi}{4}\right)\\ &= 2^{15/2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\\ &= 2^7(-1+i)\end{aligned}$$

6. $$z^4 = -1+\sqrt{3}i$$
$$= 2\left[\cos\left(\frac{2\pi}{3}+2k\pi\right)+i\sin\left(\frac{2\pi}{3}+2k\pi\right)\right]$$

Therefore,

$$\begin{aligned} z_{k=0} &= \left[2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right]^{1/4} & z_{k=1} &= \left[2\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)\right]^{1/4} \\ &= 2^{1/4}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) & &= 2^{1/4}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) \\ &= 2^{1/4}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) & &= 2^{1/4}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ z_{k=2} &= \left[2\left(\cos\frac{14\pi}{3}+i\sin\frac{14\pi}{3}\right)\right]^{1/4} & z_{k=2} &= \left[2\left(\cos\frac{20\pi}{3}+i\sin\frac{20\pi}{3}\right)\right]^{1/4} \\ &= 2^{1/4}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right) & &= 2^{1/4}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right). \\ &= 2^{1/4}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) & &= 2^{1/4}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right). \end{aligned}$$

7. (a) $$\lim_{x\to 3}\frac{3x+5}{5x-3} = \frac{3\times 3+5}{5\times 3-3} = \frac{7}{6}.$$

(b) $$\lim_{x\to 0^+}\frac{x}{|x|} = \lim_{x\to 0}\frac{x}{x} = 1.$$

(c) $$\lim_{x\to 0^-}\frac{x}{|x|} = \lim_{x\to 0}\frac{x}{-x} = -1.$$

(d) $$\lim_{x\to 0^+}\frac{x}{|x|} \neq \lim_{x\to 0^-}\frac{x}{|x|}$$

Hence, $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

8. (a) $$\lim_{x\to 2}\frac{\sqrt{(x-2)^2}}{2-x} = \lim_{x\to 2}\frac{|x-2|}{2-x}$$

Now $\lim_{x\to 2^+}\frac{|x-2|}{2-x} = \lim_{x\to 2^+}\frac{x-2}{2-x} = -1.$

But $\lim_{x\to 2^-}\frac{|x-2|}{2-x} = \lim_{x\to 2^-}\frac{-(x-2)}{2-x} = 1.$

Therefore, $\lim_{x\to 2}\frac{\sqrt{(x-2)^2}}{2-x}$ does not exist.

(b) $$\lim_{x\to\infty}\frac{2x^2+3}{-x^2+x} = \lim_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{3}{x^2}}{-\frac{x^2}{x^2}+\frac{x}{x^2}} = \lim_{x\to\infty}\frac{2+\frac{3}{x^2}}{-1+\frac{1}{x}} = \frac{2+0}{-1+0} = -2.$$

(c) $\lim_{x\to 0}\frac{\sin x - x^2 - x}{e^x - x - 1}$ is of the form $\frac{0}{0}$, so we need to apply L'Hôpital's Rule:

Hence, $\lim_{x\to 0}\frac{\sin x - x^2 - x}{e^x - x - 1} = \lim_{x\to 0}\frac{\cos x - 2x - 1}{e^x - 1}$, if the new limit exists.

This is still of the form $\frac{0}{0}$, so we apply L'Hôpital's Rule a second time:

$$\text{Hence, } \lim_{x\to 0}\frac{\sin x - x^2 - x}{e^x - x - 1} = \lim_{x\to 0}\frac{-\sin x - 2}{e^x} = \frac{0-2}{1} = -2.$$

(d) $\lim_{x\to\infty}\frac{e^x}{x^2}$ is of the form $\frac{\infty}{\infty}$, so we need to apply L'Hôpital's Rule:

$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x}$, if the new limit exists.

However, we still have the form $\frac{\infty}{\infty}$, so we re-apply L'Hôpital's Rule:

$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2}$, if the new limit exists.

But this limit does not exist, and we can't apply L'Hôpital's again; $\lim_{x\to\infty}\frac{e^x}{x^2}$ does not exist.

9. (a) Note that $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\displaystyle\lim_{x\to 0^+} \frac{1}{\sin x} = +\infty$.

Therefore, $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ yields an indeterminate form of type $\infty - \infty$.

Combining terms gives: $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$, which is of the form $\dfrac{0}{0}$.

Applying L'Hôpital's Rule twice gives:

$$\begin{aligned} \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \\ &= \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x \sin x} \\ &= \frac{0}{2} = 0. \end{aligned}$$

(b) Since $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right) = 1$, then $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$ is of the indeterminate form '1^∞'.

Let $y = \left(1 + \dfrac{1}{x}\right)^x$ and take ln of both sides to get: $\ln y = x \ln\left(1 + \dfrac{1}{x}\right) = \dfrac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$.

Now the limit, $\displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$, is of the form '$\frac{0}{0}$', so use L'Hôpital's Rule:

$$\begin{aligned} \lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\left(\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}\right)}{\left(-\frac{1}{x^2}\right)} \\ &= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x}} = 1. \end{aligned}$$

Therefore, $$\begin{aligned} \lim_{x\to\infty} y = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \\ = e^1 = e. \end{aligned}$$

10. (a) Consider

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1} e^x \\ &= e \end{aligned} \qquad \begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1} (kx^2 + 2) \\ &= k + 2 \end{aligned}$$

For f to be continuous, $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \implies k + 2 = e \implies k = e - 2$

(b) Consider

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1} (x^2 + 2x + 1) \\ &= 4 \end{aligned} \qquad \begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} (x + 1) \\ &= 2 \end{aligned}$$

For f to be continuous, $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, which is impossible.
Therefore, f cannot be continuous.