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First Name	______________
Student Number	______________
Table Number	______________

University of Wollongong

School of Mathematics and Applied Statistics

MATH 141 — MATHEMATICS 1C, PART 1

Autumn Session Examination 2007

Time Allowed: 3 hours and 15 minutes

Number of Questions: 4.

Directions to Candidates

1. Each question is to be attempted.
2. The four questions are of equal value (individual parts within a question may not be of equal value).
3. The examination paper is printed on both sides.
4. Four solution books are provided. The solution to each question is to be submitted in its own separate, clearly labelled, solution book.
5. WORKING (including all necessary reasoning) is to be shown for all solutions.
6. All notation is as used in lectures.

Examination Materials/Aids Allowed

Non-alphanumeric, non-programmable, calculators are permitted.
A one-page, double-sided, A4 size summary sheet is permitted.

Examination Materials/Aids to be supplied

A Table of Integrals is attached.

A sheet with two polar grids is attached.

This examination paper must NOT be removed from the examination room.

Question 1 (*Use a separate book for your answers to Question 1. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & 7 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix}$.

(i) Write down A^T, the transpose of the matrix A.

(ii) Compute $\det A$, the determinant of the matrix A.

(iii) Using no more than three elementary row operations, reduce the augmented matrix $(A \mid \underline{b})$ to echelon form, $\left(A^E \mid \underline{b}^E\right)$.

(iv) Hence, or otherwise, find the solution of the matrix equation $A\underline{x} = \underline{b}$.

(v) Is there a solution to $A\underline{x} = \underline{b}$ with $y = 1$? If so, write down the corresponding $\underline{x}$. If not, explain why not.

(vi) Write down the elementary matrices E_1, E_2, E_3 corresponding to the row operations performed in (iv).

(vii) Write down the simplified product of matrices EA, where $E = E_3E_2E_1$.

(b) Let

$$B = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$

(i) Write down the polynomial equation satisfied by the eigenvalues λ of B.

(ii) Hence, or otherwise, find the eigenvalues of B.

(iii) Find an eigenvector corresponding to each eigenvalue of B.

(iv) Using matrix multiplication and addition, verify that

$$B^2 + B - 6I = Z,$$

where I is the 2×2 identity matrix and Z is the 2×2 zero matrix.

(v) Multiply the equation in (iv) by B^{-1} and rearrange, to obtain down a formula for B^{-1}, *in terms of* B.

(vi) Using the formula from (b) (v), or otherwise, compute B^{-1} in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(vii) Verify that $BB^{-1} = I$.

(viii) Using the result of (b) (vi), or otherwise, solve the following systems of linear equations:

$$(I) \quad \begin{aligned} x + 2y &= 0 \\ 2x - 2y &= 0 \end{aligned}$$

$$(II) \quad \begin{aligned} x + 2y &= 3 \\ 2x - 2y &= -6 \end{aligned}$$

Question 2 (*Use a separate book for your answers to Question 2. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Simplify $\sqrt{27}\sqrt{3}$

(b) Factorise the following quadratic equation

$$y(x) = 2x^2 + x - 1.$$

(c) Sketch the graph of the function

$$xy = 4.$$

(d) Express the following as simply as possible

$$6x^3y^{-2} \times \frac{1}{24}x^{-5}y^4.$$

(e) $n!$ is the number of ways in which it is possible to arrange n objects.

Let's suppose that it takes fifteen seconds to go from one arrangement to another. Then it takes $2! \times 15 = 30$ seconds to view all the arrangements of two objects and $3! \times 15 = 90$, seconds to view all the arrangements of three objects.

How long does it take to take to view all the arrangements of ten objects? (convert your answer to days).

(f) Given that $h(0.5) = 0$ determine the zeros of the function

$$h(x) = 2x^3 - 3x^2 + 0.5.$$

(g) (i) Evaluate the following limit if it exists

$$\lim_{x \to 0} (x-2)(x-3).$$

(ii) Give an example of functions f and g such that the following hold.

$$\lim_{x \to 0} f(x) = 0, \qquad \lim_{x \to 0} g(x) = 0, \qquad \text{and} \qquad \lim_{x \to 0} \frac{f(x)}{g(x)} \qquad \text{does not exist.}$$

(iii) Evaluate the following limit if it exists

$$\lim_{x \to -\infty} \frac{2x^2 + 3x + 4}{5x^2 + 4x + 3}.$$

(h) (i) If $\sinh x = 4$, write down the *exact* value of $\cosh x$.

(ii) Does the function $f(x) = x^2 - 2x - 3$ defined on the domain $\mathbb{R}$ have an inverse? If so, find the inverse function together with its domain, range and graph. If not, *justify* your answer.

(iii) Differentiate the function $y = \cos^{-1}\left(x^2 + x\right)$.

Question 3 (*Use a separate book for your answers to Question 3. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Let $\vec{a} = (-1, 2, 2)$, $\vec{b} = (-3, 0, 4)$ and $\vec{c} = (4, 1, 3)$.

(i) Find the unit vector of $\vec{a}$ and $3\vec{a}$.

(ii) Determine if vectors $\vec{b}$ and $\vec{c}$ are perpendicular.

(iii) Find the projection of $\vec{a}$ on $\vec{b}$ and that of $\vec{a}$ on $-2\vec{b}$.

(b) (i) Find the vector parametric equation of the line $\mathcal{L}_1$ passing through the points $P(3, -1, 0)$ and $Q(1, -2, 1)$.

(ii) Given the line $\mathcal{L}$ defined by

$$\mathcal{L}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} s,$$

determine if the lines $\mathcal{L}$ and $\mathcal{L}_1$ intersect, are parallel or are skew, where $\mathcal{L}_1$ is the line found in (b)(i).

(iii) For the line $\mathcal{L}$ given in part (b)(ii), find the distance between $\mathcal{L}$ and the point $R(2, 1, 1)$.

(c) (i) Consider the three points $A(1, 1, 1)$, $B(2, -1, -2)$ and $C(3, -1, 2)$. Find $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Using (c)(i), represent the plane $\mathcal{P}_1$ that passes through the points $A(1, 1, 1)$, $B(2, -1, -2)$ and $C(3, -1, 2)$ in vector parametric form.

(iii) Using (c)(i), show that the linear form of the equation representing $\mathcal{P}_1$, found in part (c)(ii), is given by

$$8x + 7y - 2z = 13.$$

Question 4 (*Use a separate book for your answers to Question 4. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) (i) Differentiate the function $y = \frac{1}{5}\left(7x^3 - \frac{1}{2}x^2 + 2x - 3\right)$ with respect to x

(ii) Differentiate the function $y = x \tan x$ with respect to x.

(iii) By writing $\operatorname{sech} x$ as $(\cosh x)^{-1}$ show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x.$$

(iv) Differentiate the function $y = \cos(\sin x)$ with respect to x.

(v) Differentiate the function $y = x^{\sin x}$.

(b) Find $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ if $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + \sin x}{2y - \sin y}$.

(c) A curve is defined by the parametric equation $x(t) = t\cos(t)$ and $y(t) = \sin(t)$. Calculate $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

(d) (i) Find the value(s) of r when $\theta = \frac{6.5\pi}{12}$ if $r = 5\cos 3\theta$.

(ii) Using the polar graph paper provided at the end of the exam paper sketch the function $r = 5\cos 3\theta$, $0 \le \theta \le 2\pi$. The following set of values may be useful

θ	$\frac{0\pi}{12}$	$\frac{1\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$	$\frac{7\pi}{12}$	$\frac{8\pi}{12}$	$\frac{9\pi}{12}$	$\frac{10\pi}{12}$	$\frac{11\pi}{12}$	$\frac{12\pi}{12}$	$\frac{13\pi}{12}$
r	5	3.5	0	-3.5	-5	-3.5	0	3.5	5	3.5	0	-3.5	-5	-3.5
θ	$\frac{14\pi}{12}$	$\frac{15\pi}{12}$	$\frac{16\pi}{12}$	$\frac{17\pi}{12}$	$\frac{18\pi}{12}$	$\frac{19\pi}{12}$	$\frac{20\pi}{12}$	$\frac{21\pi}{12}$	$\frac{22\pi}{12}$	$\frac{23\pi}{12}$	$\frac{24\pi}{12}$			
r	0	3.5	5	3.5	0	-3.5	-5	-3.5	0	3.5	5			

You should also use your answer to (d)(i).
You should detach the polar graph paper from the exam paper and put it inside your answer book. Failure to do this may mean that your answer is not marked.

(e) Evaluate the following integrals.

(i) $\int_2^3 3e^{2x}\,\mathrm{d}x$

(ii) $\int e^{18x} \sin 4x\,\mathrm{d}x$

(iii) $\int \frac{\mathrm{d}x}{x\sqrt{19x+6}}$

(iv) $\int_2^4 (2x-3)^5\,\mathrm{d}x$

(v) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{75}^{\tan x} t^2 \sin t\,\mathrm{d}t$