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First Name	______________
Student Number	______________
Table Number	______________

University of Wollongong

School of Mathematics and Applied Statistics

MATH 141 — MATHEMATICS 1C, PART 1

Autumn Session Examination 2006

Time Allowed: 3 hours and 15 minutes

Number of Questions: 4.

Directions to Candidates

1. Each question is to be attempted.
2. The four questions are of equal value (individual parts within a question may not be of equal value).
3. The examination paper is printed on both sides.
4. Four solution books are provided. The solution to each question is to be submitted in its own separate, clearly labelled, solution book.
5. WORKING (including all necessary reasoning) is to be shown for all solutions.
6. All notation is as used in lectures.

Examination Materials/Aids Allowed

Non-alphanumeric, non-programmable, calculators are permitted.
A one-page, double-sided, A4 size summary sheet is permitted.

Examination Materials/Aids to be supplied

A Table of Integrals is attached.

A sheet with two polar grids is attached.

This examination paper must NOT be removed from the examination room.

Question 1 (*Use a separate book for your answers to Question 1. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Let $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & -1 \\ 4 & -3 & -1 \end{pmatrix}$, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$.

(i) Find the transpose A^T of the matrix A.

(ii) Calculate the determinant of the matrix A.

(iii) Does the matrix A have an inverse? Justify your answer.

(iv) Using four elementary row operations, reduce the augmented matrix $\left(A \mid \underset{\sim}{b}\right)$ to echelon form, $\left(A^E \mid \underset{\sim}{b}^E\right)$.

(v) Hence, or otherwise, find the solution of the matrix equation $A\underset{\sim}{x} = \underset{\sim}{b}$.

(vi) Write down the elementary matrices E_1, E_2, E_3, E_4 corresponding to the row operations performed in (iv).

(b) Let $B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

(i) Compute $\det(B - \lambda I)$ and write down the characteristic equation for B.

(ii) Find the eigenvalues of B.

(iii) Find the eigenvector corresponding to each eigenvalue of B.

(iv) Using matrix multiplication and addition compute the value of

$$B^2 - 6B + 5I$$

where I is the 2×2 identity matrix.

(v) Find the inverse of the matrix B.

(vi) Using your previous answer, or otherwise, solve the following system of linear equations:

$$\begin{aligned} 3x + 2y &= 1 \\ 2x + 3y &= -1. \end{aligned}$$

Question 2 (*Use a separate book for your answers to Question 2. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Calculate the average speed in kilometres per hour of a person who walks 35.2 metres in one minute.

(b) Simplify

$$\frac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}.$$

(c) (i) If $0 \leq x \leq 2\pi$, find the value(s) of x which satisfy the equation

$$\sin(x) = \frac{1}{2}.$$

(ii) Evaluate exactly

$$\sin^{-1}\frac{1}{2}.$$

(iii) Use the half-angle formula to write down the exact value of $\sin\frac{\pi}{8}$.

(d) Determine the zeros of the function

$$h(x) = x^3 - 7x + 6.$$

(e) (i) Find the exact value of $\tanh^{-1} 0.5$.

(ii) Graph the function $y = \cosh x$. From your graph identify the domain and range of the inverse function $y = \cosh^{-1} x$.

(f) (i) If $\lim_{x\to c} f(x) = 2$ and $\lim_{x\to c} g(x) = 5$ find

$$\lim_{x\to c}\frac{f(x)}{3g(x)}.$$

(ii) Give an example of functions f and g such that the following hold.

$$\lim_{x\to 0} f(x) = 0, \qquad \lim_{x\to 0} g(x) = 0, \qquad \text{and} \qquad \lim_{x\to 0}\frac{f(x)}{g(x)} = 4.$$

(iii) Evaluate the following limit if it exists

$$\lim_{x\to\infty}\frac{4x^2 + x}{x^2 + 2x + 1}.$$

(g) (i) Let f and g be the functions given by

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{x} - 1.$$

Find $h(x) = (f \circ g)(x)$. What are the domain and range of this function?

(ii) Differentiate the function $y = \sinh^{-1} x^2$.

Question 3 (*Use a separate book for your answers to Question 3. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) Consider the three vectors $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.

(i) Find the angle (in exact form) between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(ii) Find the vector of length three units acting in the opposite direction to $\underset{\sim}{b}$.

(iii) Determine whether $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are perpendicular.

(iv) Find the vector projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.

(v) Find the area of the parallelogram with adjacent sides formed by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(vi) Find the value of $\left[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}\right]$ and give a geometric interpretation of the result.

(b) Consider

- the point $A\,(1, -2, 7)$,
- the straight line

$$\mathcal{L}_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} t, \quad t \in \mathbb{R},$$

- the straight line $\mathcal{L}_2 : \dfrac{x-2}{2} = \dfrac{y-3}{1} = \dfrac{z+2}{1}$,
- the plane $\mathcal{P}_1 : 2x + y - z = 3$,
- and the plane $\mathcal{P}_2 : x + 2y + z = 3$.

(i) Find the equation of the line $\mathcal{L}_3$ which is parallel to $\mathcal{L}_1$ and passes through A.

(ii) Find the angle (in exact form) which $\mathcal{L}_1$ makes with the positive z-axis.

(iii) Find the intersection point (if it exists) of $\mathcal{L}_1$ with $\mathcal{P}_1$.

(iv) Find the distance from A to $\mathcal{L}_1$.

(v) Find the linear form of the equation of the plane $\mathcal{P}_3$ which contains A and $\mathcal{L}_1$.

(vi) Find the angle of intersection between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

(vii) Find the equation of $\mathcal{L}_4$, the line of intersection between the planes $\mathcal{P}_1$ and $\mathcal{P}_2$.

(viii) Determine whether $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel, intersecting or skew lines.

(c) A system of three forces $\mathbf{F}_1\,250\,\angle 105^o, \mathbf{F}_2\,100\,\angle 200^o, \mathbf{F}_3\,300\,\angle 30^o$ (in Newtons) act on a single point with directions as stated. Using horizontal and vertical components, find the magnitude and direction of the resultant force which could replace the system.

Question 4 (*Use a separate book for your answers to Question 4. Failing to use a separate answer book may mean that your answer is not marked.*)

(a) (i) The position x of a ball as a function of time t is given by

$$x = At - \frac{1}{B}\ln\left(1 + Ce^{-Dt}\right) + E,$$

where A, B, C, D and E are constants. By differentiating this expression find the velocity (v) of the ball, where $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(ii) Differentiate the function $y = \cosh x \sinh x$ with respect to x.

(iii) Show that if $y = \cot x$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}^2 x.$$

(b) If y is given implicitly by the equation $y^2 + y = \sin x$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(c) A curve is defined by the parametric equation $x(t) = \sin\left(t^2\right)$ and $y(t) = \cos(t)$. Calculate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(d) (i) Find the value(s) of r when $\theta = \dfrac{6.5\pi}{12}$ if $r^2 = -8\sin 2\theta$.

(ii) Using the polar graph paper provided at the end of the exam paper sketch the function $r^2 = -8\sin 2\theta$, $\dfrac{\pi}{2} \leq \theta \leq \pi$, $\dfrac{3\pi}{2} \leq \theta \leq 2\pi$. The following set of values may be useful

θ	$\frac{6\pi}{12}$	$\frac{7\pi}{12}$	$\frac{8\pi}{12}$	$\frac{9\pi}{12}$	$\frac{10\pi}{12}$	$\frac{11\pi}{12}$	π	$\frac{18\pi}{12}$	$\frac{19\pi}{12}$	$\frac{20\pi}{12}$	$\frac{21\pi}{12}$	$\frac{22\pi}{12}$	$\frac{23\pi}{12}$	2π
r	0	2	2.6	2.8	2.6	2	0	0	2	2.6	2.8	2.6	2	0

You should also use your answer to (d)(i).

(e) Evaluate the following integrals.

(i) $\displaystyle\int \left(\frac{12}{x} + x^5\right) \mathrm{d}x$

(ii) $\displaystyle\int \frac{1}{x\sqrt{10 - x^2}} \mathrm{d}x$

(iii) $\displaystyle\int_e^{3e} \sqrt{x^2 + e^2}\, \mathrm{d}x$

(iv) $\displaystyle\int \frac{1}{c\ln c} \mathrm{d}c$

(v) $\displaystyle\int \tan x\, \mathrm{d}x$ (use a substitution)