MATH141 – Autumn 2007
Outline Solutions to Tutorial Sheet – Week 2

1.

$$(a)\quad \sum_{\alpha=1}^{2}(2\alpha+1) = 3+5 = 8 \qquad (b)\quad \sum_{\alpha=1}^{2} 2\alpha+1 = (2+4)+1 = 7 \qquad (c)\quad \sum_{i=1}^{2} 2\alpha+1 = (2\alpha+2\alpha)+1 = 4\alpha+1$$

Hence only the summation in part (b) is equal to 7.

2.

$$(a)\qquad \sum_{k=1}^{4} k\sin\frac{k\pi}{2} = \sin\frac{\pi}{2} + 2\sin\pi + 3\sin\frac{3\pi}{2} + 4\sin 2\pi$$
$$= 1+0-3+0$$
$$= -2.$$

$$(b)\qquad \sum_{k=0}^{4} 1 = 1+1+1+1+1 = 5.$$

$$(c)\quad \sum_{k=1}^{20}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{20}-\frac{1}{21}\right)$$
$$= 1-\frac{1}{21}$$
$$= \frac{20}{21}$$

3.

$$\sum_{k=1}^{6}\left(4k^3-2k+1\right) = 4\sum_{k=1}^{6}k^3 - 2\sum_{k=1}^{6}k + \sum_{k=1}^{6}1$$

$$\text{Now}\sum_{k=1}^{6}k^3 = \left(\frac{6\,(6+1)}{2}\right)^2 = (21)^2,$$

$$\sum_{k=1}^{6}k = \frac{6\,(6+1)}{2} = 21,$$

$$\sum_{k=1}^{6}1 = 6.$$

$$\text{Thus}\sum_{k=1}^{6}\left(4k^3-2k+1\right) = 4\,(21)^2 - 2\,(21) + 6$$
$$= 4\times 441 - 42 + 6$$
$$= 1728.$$

4.

$$\begin{aligned}
\sum_{k=1}^{10} (k+2)^3 &= \sum_{k=1}^{10} \left(k^3 + 6k^2 + 12k + 8\right), \\
&= \sum_{k=1}^{k=10} k^3 + 6\sum_{k=1}^{k=10} k^2 + 12\sum_{k=1}^{k=10} k + 8\sum_{k=1}^{k=10} 1,
\end{aligned}$$

$$\text{Now,} \quad \sum_{k=1}^{k=10} k^3 = \frac{(10)^2\,(11)^2}{4} = 3025,$$

$$\sum_{k=1}^{k=10} k^2 = \frac{(10)\,(11)\,(21)}{6} = 385,$$

$$\sum_{k=1}^{k=10} k = \frac{(10)\,(11)}{2} = 55,$$

$$\text{and} \quad \sum_{k=1}^{k=10} 1 = 10,$$

$$\begin{aligned}
\text{Thus,} \quad \sum_{k=1}^{10} (k+2)^3 &= 3025 + 6\,(385) + 12\,(55) + 8\,(10) \\
&= 6075.
\end{aligned}$$

5. (a)

$$\begin{aligned}
\sum_{i=1}^{4}\sum_{j=1}^{4} \delta_{ij}\delta_{j3}x_i x_3 &= \sum_{i=1}^{4} \delta_{i3}x_i x_3 \qquad \text{(replacing } j \text{ by 3)} \\
&= x_3 x_3 \qquad \text{(replacing } i \text{ by 3)} \\
&= (x_3)^2
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{k=1}^{4}\sum_{m=1}^{2} \delta_{km}\delta_{k3} &= \sum_{k=1}^{4} (\delta_{k1}\delta_{k3} + \delta_{k2}\delta_{k3}) \\
&= \sum_{k=1}^{4} (0+0) \qquad (k \text{ can't be 1 and 3 at the same time}) \\
&= 0.
\end{aligned}$$

6. (a) First $\sum_{j=1}^{4}\sum_{i=1}^{4} \delta_{i3}\delta_{ij} = \sum_{i=1}^{4} \delta_{i3}\delta_{i1} + \sum_{i=1}^{4} \delta_{i3}\delta_{i2} + \sum_{i=1}^{4} \delta_{i3}\delta_{i3} + \sum_{i=1}^{4} \delta_{i3}\delta_{i4}$.

Dealing with the first term on the right-hand side, we have

$\sum_{i=1}^{4} \delta_{i3}\delta_{i1} = \delta_{13}\delta_{11} + \delta_{23}\delta_{21} + \delta_{33}\delta_{31} + \delta_{43}\delta_{41} = 0.$

Similarly, the second and fourth terms are also 0.

For the third term, we have

$\sum_{i=1}^{4} \delta_{i3}\delta_{i3} = \delta_{13}\delta_{13} + \delta_{23}\delta_{23} + \delta_{33}\delta_{33} + \delta_{43}\delta_{43} = 1,$

since the third term equals 1 (and all the others are 0). Therefore, the original expression equals 1.

(b) We have $\sum_{j=1}^{3}\sum_{i=1}^{3}(3i+j)\,\delta_{ij}=\sum_{i=1}^{3}(3i+1)\,\delta_{i1}+\sum_{i=1}^{3}(3i+2)\,\delta_{i2}+\sum_{i=1}^{3}(3i+3)\,\delta_{i3}$

For the terms on the right-hand side,

$$\sum_{i=1}^{3}(3i+1)\,\delta_{i1}=(3+1)\,\delta_{11}+(6+1)\,\delta_{21}+(9+1)\,\delta_{31}=(3+1)=4.$$
$$\sum_{i=1}^{3}(3i+2)\,\delta_{i2}=(3+2)\,\delta_{12}+(6+2)\,\delta_{22}+(9+2)\,\delta_{32}=(6+2)=8.$$
$$\sum_{i=1}^{3}(3i+3)\,\delta_{i3}=(3+3)\,\delta_{13}+(6+3)\,\delta_{23}+(9+3)\,\delta_{33}=(9+3)=12.$$

So the original expression equals $4+8+12=24$.

7.

(a)
$$\begin{aligned}2^{5x}\times 8^{x}\div 4^{2x}&=2^{5x}\times 2^{3x}\div 2^{4x}\\&=2^{5x+3x-4x}\\&=2^{4x}\end{aligned}$$

(b)
$$\begin{aligned}&\left(x^2-y^2\right)^{1/2}\times(x-y)^{3/2}\times(x+y)^{-1/2}\\&=(x-y)^{1/2}(x+y)^{1/2}\times(x-y)^{3/2}\times(x+y)^{-1/2}\\&=(x-y)^{1/2+3/2}(x+y)^{1/2-1/2}\\&=(x-y)^2\end{aligned}$$

(c)
$$\begin{aligned}7\log 5-\log 25&=7\log 5-\log 5^2\\&=7\log 5-2\log 5\\&=5\log 5\end{aligned}$$

(d)
$$\begin{aligned}\frac{\log 64}{\log 16}&=\frac{\log 2^6}{\log 2^4}\\&=\frac{6\log 2}{4\log 2}\\&=\frac{3}{2}\end{aligned}$$

(e)
$$\begin{aligned}2\sqrt{10}\times 4\sqrt{15}&=8\sqrt{2}\times\sqrt{5}\times\sqrt{3}\times\sqrt{5}\\&=40\sqrt{6}.\end{aligned}$$

(f)
$$\begin{aligned}3\sqrt{32}+2\sqrt{50}-8\sqrt{18}&=12\sqrt{2}+10\sqrt{2}-24\sqrt{2}\\&=-2\sqrt{2}\end{aligned}$$

8.

(a) Since $25=5^2$ and $\sqrt{5}=5^{1/2}$, we have
$$\begin{aligned}x&=\log_5 25\sqrt{5}\\&=\log_5 5^2 5^{1/2}\\&=\log_5 5^{5/2}\\&=\frac{5}{2}.\end{aligned}$$

(b) We have $\log_5 x^7=7\log_5 x$
$$\begin{aligned}&\Rightarrow 3\log_5 x+4=7\log_5 x\\&\Rightarrow 4\log_5 x=4\\&\Rightarrow \log_5 x=1\\&\Rightarrow x=5\end{aligned}$$

9.

(a) $3x^2+4x-7=(3x+7)(x-1)$

(b)
$$\begin{aligned}a^2-b^2+2a-2b&=(a-b)(a+b)+2(a-b)\\&=(a-b)(a+b+2)\end{aligned}$$

10.

$$\begin{aligned}5x^2-26x+24&=0\\&\Rightarrow(5x-6)(x-4)=0\\&\Rightarrow 5x-6=0\quad\text{or}\quad x-4=0\\&\Rightarrow x=\frac{6}{5}\quad\text{or}\quad x=4.\end{aligned}$$

11.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4 \times 4 \times 2}}{2 \times 4} \\ &= \frac{-11 \pm \sqrt{121 - 32}}{8} \\ &\approx \frac{-11 \pm 9.434}{8} \end{aligned}$$

Therefore, $x \approx -0.196$ or -2.554

12.

$$\begin{aligned} 9^x - 10\left(3^x\right) + 9 &= 0. \\ \Rightarrow 3^{2x} - 10\left(3^x\right) + 9 &= 0 \\ \text{Let } u &= 3^x \\ \text{Then,} \quad u^2 - 10u + 9 &= 0 \\ \Rightarrow (u-1)(u-9) &= 0 \\ \Rightarrow u = 1 \quad &\text{or} \quad u = 9 \\ \text{Therefore,} \quad 3^x = 1 \quad &\text{or} \quad 3^x = 9 \\ \Rightarrow x = 0 \quad &\text{or} \quad x = 2 \end{aligned}$$

13.

$$(a) \quad \frac{3x^3}{4a^2} \times \frac{ay - a}{xy^2} \div \frac{3y - 3}{4ay^2}$$

$$= \frac{3x^3}{4a^2} \times \frac{a\,(y-1)}{xy^2} \times \frac{4ay^2}{3\,(y-1)}$$

$$= x^2$$

$$(b) \quad \frac{5x}{3} - \frac{2x+3}{4} + \frac{x}{6}$$

$$= \frac{20x}{12} - \frac{3\,(2x+3)}{12} + \frac{2x}{12}$$

$$= \frac{20x - 6x - 9 + 2x}{12}$$

$$= \frac{16x - 9}{12}$$