Family Name . . . .
First Name . . . .
Student Number . . . .
Table Number . . . .

UNIVERSITY OF WOLLONGONG

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH141 - MATHEMATICS 1C, PART 1

Autumn Session Examination 2001

Time Allowed: 3 hours 15 minutes

Number of Questions: 4

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The four questions are of equal value (individual parts within a question may not be of equal value).
3. The examination paper is printed on both sides.
4. Four solution books are provided. The solution to each question is to be submitted in its own separate, clearly labelled solution book.
5. WORKING (including all necessary reasoning) is to be shown for all solutions.
6. All notation is as used in lectures.
7. A Table of Integrals is attached.

EXAMINATION MATERIALS/AIDS ALLOWED

Non-alphanumeric, non-programmable calculators are permitted.
A one-page, double-sided, A4 size summary sheet is permitted.

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED

None.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 *(Use a separate book for your answers to Question 1.)*

(a) Consider the system of equations $A\underset{\sim}{x} = \underset{\sim}{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

(i) Row reduce the augmented matrix $(A|I|\underset{\sim}{b})$ to solve for $\underset{\sim}{x}$ and to find A^{-1}.

(ii) Verify that $\underset{\sim}{x} = A^{-1}\underset{\sim}{b}$.

(iii) Write down the elementary matrices corresponding to the row operations performed in part (a)(i), which put A in reduced row echelon form.
Write out an expression for A^{-1} in terms of these elementary matrices. (Do ***not*** multiply the elementary matrices together.)

(b) Give brief answers (a few lines) to the following. Show working, if required.

(i) Evaluate $\displaystyle\sum_{i=1}^{10}\sum_{j=1}^{20} \delta_{ij}\,(2\delta_{ij} + \delta_{i+2,j})$.

(ii) If $A^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$, evaluate $(AB)^{-1}$ and $(A^T)^{-1}$.

(iii) Find the determinant of $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 8 & 2 & 0 \end{pmatrix}$.

(iv) Let A and B be $n \times n$ matrices. Explain why

$$(A+B)^2 = A^2 + 2AB + B^2$$

is not, in general, valid.

(v) Let A be a 3×3 matrix. Show that the matrix $B = A + A^T$ is symmetric.

(vi) Consider the augmented matrix

$$\left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right).$$

Show that the system has non-trivial solutions when $ad - bc = 0$. You can assume that $a \neq 0$.

Question 2 *(Use a separate book for your answers to Question 2.)*

(a) Simplify as far as possible the expression

$$\frac{\sqrt{\cot^2\theta + 1}\sqrt{1 - \sin^2\theta}}{\sqrt{\sec^2\theta - 1}}.$$

(b) (i) Show that $(x - 5)$ is a factor of $f(x) = 2x^3 - 11x^2 + 2x + 15$.

(ii) Using long division, or otherwise, find all the factors of $f(x)$.

(c) Solve for x in the equation $\sin x + \sqrt{3}\cos x = 0, \quad 0 \leq x \leq 2\pi$.
Give your answers in terms of π.

(d) Let $f(x) = \dfrac{x - 2}{x + 1}$.

(i) Sketch $f(x)$.

(ii) Write down the largest domain and range of $f(x)$.

(iii) Does the inverse function, $f^{-1}(x)$, exist? If so, find an expression for f^{-1}. If not, explain why.

(e) Prove that $\cosh^2 x - \sinh^2 x = 1$.

(f) Find the following limits, if they exist.

(i) $\displaystyle\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 9}$

(ii) $\displaystyle\lim_{x \to \pi/2} \frac{\sin 2x}{\cos x}$

(iii) $\displaystyle\lim_{x \to 4} \frac{x^3 - 4x^2 + 9x - 36}{x^2 + 5}$

(iv) $\displaystyle\lim_{x \to \infty} \frac{3x^4 - x^2 + 2}{5x^4 + x^3 + 1}$

(v) $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 3x + 4}{x - 1}$

Question 3 *(Use a separate book for your answers to Question 3.)*

(a) Let $\underset{\sim}{a} = (2, -1, 3)$ and $\underset{\sim}{b} = (1, 2, 2)$. Compute the following.

(i) $5\underset{\sim}{b} - 3\underset{\sim}{a}$

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$

(iii) The angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(iv) $\underset{\sim}{a} \times \underset{\sim}{b}$

(v) The component of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (that is, the length of the projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$).

(b) Let $P = (1, 5, 1)$ and $Q = (2, 7, -1)$.

(i) Write down the equation of the line, $\mathcal{L}_1$, through P and Q.

(ii) Find the distance of the line $\mathcal{L}_1$, found in part (b)(i), from the origin $(0, 0, 0)$.

(iii) Find the equation of the line, $\mathcal{L}_2$, which passes through $R = (2, 4, 5)$ and is parallel to $\underset{\sim}{a} = (3, 0, 1)$.

(v) Determine whether the lines $\mathcal{L}_1$, found in part (b)(i), and $\mathcal{L}_2$, found in part (b)(iii), intersect, are parallel or are skew. If the lines intersect, find the point of intersection. If the lines are skew, find the distance between them.

(c) Let $A = (1, 1, 1)$, $B = (-3, 0, 3)$ and $C = (7, 1, -6)$.

(i) Find an equation representing the plane, $\mathcal{P}_1$, through A, B and C.

(ii) Find a vector $\underset{\sim}{n}$ normal to the plane $\mathcal{P}_1$, found in part (c)(i).

(iii) Find the equation of the line, $\mathcal{L}_3$, which passes through the point $(-1, 0, 1)$ and is perpendicular to the plane $\mathcal{P}_1$, found in part (c)(i).

(iv) Find the intersection of $\mathcal{P}_1$, found in part (c)(i), with the plane $\mathcal{P}_2$ whose equation is $x - 2y + z = 3$.

(d) Interpret geometrically the result

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c} = 0.$$

Question 4 *(Use a separate book for your answers to Question 4.)*

(a) (i) Simplify $\cos(\sin^{-1} x)$ for $|x| \le 1$.

(ii) Show that $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$.

(b) Differentiate the following functions with respect to x.

(i) $x\cosh(x^2)$

(ii) $\dfrac{\sin x}{2+\cos x}$

(iii) $\ln\sqrt{x^2+1}$

(iv) x^{x^2}

(v) $\displaystyle\int_1^{x^2} \sin(\cos t)\ dt$

(c) If y is given implicitly by the equation $xe^{\sin y} = e^y$, find $\dfrac{dy}{dx}$ in terms of x and y.

(d) A curve is defined by the parametric equations

$$x = \sin t \quad \text{and} \quad y = \cos t.$$

Calculate $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

(e) (i) A point is given in polar coordinates by $\left(-1, -\frac{3\pi}{4}\right)$.

A. Give another polar representation of this point with $r > 0$.

B. Find the Cartesian coordinates of this point.

(ii) A curve is described by the Cartesian equation

$$x^2 + y^2 = 2xy.$$

Express the equation of the curve in its *simplest* polar form.

(f) Evaluate the following integrals.

(i) $\displaystyle\int \left(\cos^2 x - \sin^2 x\right)\ dx$

(ii) $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$

(iii) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{x^2+1}}$

(iv) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\ d\theta$

(v) $\displaystyle\int \frac{(\ln x)^2}{x}\ dx$

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TABLE OF INTEGRALS

[1] $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1$$

[2] $$\int \frac{dx}{x} = \ln|x| + c$$

[3] $$\int e^x\,dx = e^x + c$$

[4] $$\int \sin x\,dx = -\cos x + c$$

[5] $$\int \cos x\,dx = \sin x + c$$

[6] $$\int \tan x\,dx = \ln|\sec x| + c$$

[7] $$\int \sec^2 x\,dx = \tan x + c$$

[8] $$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

[9] $$\int \sinh x\,dx = \cosh x + c$$

[10] $$\int \cosh x\,dx = \sinh x + c$$

[11] $$\int \tanh x\,dx = \ln(\cosh x) + c$$

[12] $$\int x(ax+b)^n\,dx = \frac{1}{a^2}(ax+b)^{n+1}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + c, \quad n \neq -1, -2$$

[13] $$\int \frac{x^2}{ax+b}\,dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\ln|ax+b|\right] + c$$

[14] $$\int \frac{x^2}{(ax+b)^2}\,dx = \frac{1}{a^3}\left[ax+b - \frac{b^2}{ax+b} - 2b\ln|ax+b|\right] + c$$

[15] $$\int x\sqrt{ax+b}\,dx = \frac{2}{a^2}\left[\frac{(ax+b)^{5/2}}{5} - \frac{b(ax+b)^{3/2}}{3}\right] + c$$

[16] $$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{2ax-4b}{3a^2}\sqrt{ax+b} + c$$

[17] $$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c, \quad b > 0$$

[18] $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

[19] $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

[20] $$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$$

[21] $$\int \frac{1}{(a^2-x^2)^2}\,dx = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right| + c$$

[22] $$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + c$$

[23] $$\int \frac{1}{(a^2-x^2)^{3/2}}\,dx = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + c$$

[24] $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + c$$

[25] $$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + c$$

[26] $$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right| + c$$

[27] $$\int \frac{1}{(x^2\pm a^2)^{3/2}}\,dx = \pm\frac{1}{a^2}\frac{x}{\sqrt{x^2\pm a^2}} + c$$

[28] $$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| + c$$

[29] $$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right| + c$$

[30] $$\int \frac{1}{b+ke^{ax}}\,dx = \frac{1}{ab}\left[ax - \ln(b+ke^{ax})\right] + c, \quad ab \neq 0$$

[31] $$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + c$$

[32] $$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx) + c$$

[33] $$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

[34] $$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

[35] $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

[36] $$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

[37] $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

[38] $$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

[39] $$\int x^n \sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

[40] $$\int x^n \cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$$