Family Name . . . .
First Name . . . .
Student Number . . . .
Table Number . . . .

UNIVERSITY OF WOLLONGONG
SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH141 - MATHEMATICS 1C, PART 1

Autumn Session Examination 2003

Time Allowed: 3 hours 15 minutes

Number of Questions: 4

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The four questions are of equal value (individual parts within a question may not be of equal value).
3. The examination paper is printed on both sides.
4. Four solution books are provided. The solution to each question is to be submitted in its own separate, clearly labelled solution book.
5. WORKING (including all necessary reasoning) is to be shown for all solutions.
6. All notation is as used in lectures.
7. A Table of Integrals is attached.

EXAMINATION MATERIALS/AIDS ALLOWED

Non-alphanumeric, non-programmable calculators are permitted.
A one-page, double-sided, A4 size summary sheet is permitted.

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED

None.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1 *(Use a separate book for your answers to Question 1.)*

(a) Consider the system of equations $A\underset{\sim}{x} = \underset{\sim}{b}$, where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 0 \\ -1 & 3 & -3 \end{pmatrix}, \quad \underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{pmatrix} -6 \\ -7 \\ 11 \end{pmatrix}.$$

(i) Using no more than three row operations, reduce the augmented matrix $(A|\underset{\sim}{b})$ to echelon form, $(A^E|\underset{\sim}{b}^E)$.

(ii) Find the solution, $\underset{\sim}{x}$, if it exists.

(iii) Write down the elementary matrices, E_1, E_2, E_3, corresponding to the row operations performed in part (a)(i).

(iv) Find the matrix $E = E_3E_2E_1$, the product of elementary matrices.

(v) Verify that $A^E = EA$, where A^E is the row echelon form of A found in part (a)(i).

(b) (i) Find an expression for the determinant $\begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 + x \end{vmatrix}$.

(ii) Let A and B be $n \times n$ matrices, $n \in$. Explain why the expression
$$(A + B)^2 = A^2 + 2AB + B^2$$
is not, in general, valid.

(iii) Find the values of $a \in \mathbb{R}$ for which the following system has non-trivial solutions.
$$\left(\begin{array}{cc|c} 1 & a & 0 \\ 2a & 1 + 3a & 0 \end{array}\right)$$

(iv) Simplify the summation $\displaystyle\sum_{j=1}^{100}\sum_{i=1}^{100}\sum_{k=1}^{100} \delta_{ij}\delta_{ki}\delta_{j2}a_{ij}a_{ki}$.

(v) Let $A^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Find $(A^T)^{-1}$ and $(A^2)^{-1}$.

Question 2 *(Use a separate book for your answers to Question 2.)*

(a) (i) Simplify the fraction $\dfrac{\sqrt{5}}{\sqrt{45}}$.

(ii) Solve the following equation for m.

$$m\,(m-2)=\tfrac{1}{2}$$

(iii) Simplify the following expression as much as possible.

$$\ln x^2+\ln y-\ln x-\ln y^2$$

(iv) Solve the the following equation for y.

$$\frac{1}{\sqrt{y}}-\sqrt{y}=-\frac{1}{x}$$

(v) If $f(x)=x^2$ and $g(z)=\sin z$ find $(f\circ g)(a)$ and $(g\circ f)(x)$.

(vi) Use the Remainder Theorem to find one linear factor of x^3+5x^2+7x+2.
Use polynomial division to find the corresponding quadratic factor.

(vii) Find the exact values for $\cos\theta, \sin\theta$, $\tan\theta$ and $\sec\theta$ when $\theta=\dfrac{17\pi}{6}$.

(b) Let $f(x)=\dfrac{1}{x+1}$ be a real valued function.

(i) Sketch the function $y=f(x)$.

(ii) State the domain and range of $f(x)$.

(iii) Find the inverse $f^{-1}(x)$, if it exists.

(c) (i) Write down the definition of $\cosh x$.
Hence evaluate $\cosh(1.5)$ correct to 3 decimal places.

(ii) Sketch the curve $y=\cosh x$. State the domain and range of the function $\cosh x$, explaining how these are related to your sketch.

(iii) Referring to your sketch from part (c)(ii), explain why it is necessary to restrict the domain of the function $f(x)=\cosh x$ in order to define the inverse function $y=f^{-1}(x)=\cosh^{-1}x$.

(d) (i) Find a polar representation for the point whose Cartesian coordinates are $(5,-12)$.

(ii) Sketch the graph of the polar equation $\theta=\dfrac{\pi}{3}$ and find the Cartesian form of the equation.

(iii) Sketch the graph of the Cartesian equation $y^2=4x$ and find the polar form of the equation.

Question 3 *(Use a separate book for your answers to Question 3.)*

(a) Let $\utilde{a} = (3, -1, 2)$, $\utilde{b} = (5, 2, -1)$ and $\utilde{c} = (4, 3, -1)$.

(i) Find the unit vector of $2\utilde{c} - 3\utilde{a}$.

(ii) Determine if the vectors $\utilde{a}$ and $\utilde{c}$ are perpendicular.

(iii) Find $\utilde{a} \bullet \utilde{c} \times \utilde{b}$ and give a geometrical interpretation of your result.

(iv) Find the component of $\utilde{a}$ on $-\utilde{b}$.

(b) (i) Find the equation of the line, $\mathcal{L}_1$, passing through the points $(1, 1, 1)$ and $(0, -1, 1)$.

(ii) Find the equation of the line, $\mathcal{L}_2$, passing through the point $(1, -1, 2)$ and parallel to $\utilde{a} = (-2, 1, -1)$.

(iii) Determine whether the lines $\mathcal{L}_1$ and $\mathcal{L}_2$, found in parts (b)(i) and (b)(ii) respectively, intersect, are parallel or are skew.
If the lines intersect, find the point of intersection. If the lines are skew, find the distance between them.

(iv) Find the distance between the line $\mathcal{L}_1$, found in part (b)(i), and the point $(1, -1, 2)$. Does this distance represent the distance between the lines $\mathcal{L}_2$, found in part (b)(ii), and $\mathcal{L}_1$? Give reasons for your answer.

(c) (i) Find an equation, in vector parametric form, representing the plane $\mathcal{P}_1$ passing through the points $(2, -1, 0)$, $(3, -2, -2)$ and $(1, -1, 1)$.

(ii) Show that the linear form of the equation representing $\mathcal{P}_1$, found in part (c)(i), is given by $x - y + z = 3$.

(iii) Let the linear form of the equation representing the plane $\mathcal{P}_2$ be given by

$$x - 2y + z - 3 = 0.$$

Determine whether the planes $\mathcal{P}_1$, found in part (c)(i), and $\mathcal{P}_2$ intersect or are parallel. If the two planes intersect, find the equation of the intersection line. If the two planes are parallel, find the distance between them.

(iv) Find the distance between the plane $\mathcal{P}_2$, from part (c)(iii), and the point $(1, 3, -1)$.

Question 4 *(Use a separate book for your answers to Question 4.)*

(a) Evaluate the following limits, if they exist.

(i) $\displaystyle\lim_{x\to 0}\frac{1}{x}$

(ii) $\displaystyle\lim_{x\to 1}\frac{x^2+3x-4}{x^2-x}$

(iii) $\displaystyle\lim_{x\to\infty}\frac{5x^3+7x^2+9x+11}{13x^2+15x+17}$

(iv) $\displaystyle\lim_{x\to-\infty}\frac{7x^2+19x+37}{45x^2+1937x+\pi}$

(b) Differentiate the following functions with respect to x, showing all your working.

(i) e^{3x}

(ii) $x^2\sinh x$

(iii) $\displaystyle\frac{\sin x}{x^2}$

(iv) $\sqrt{x^2-1}$

(v) $\sin^{-1}(2x)$

(c) If y is given implicitly by the equation $xy^2+y=\cosh x$ find $\dfrac{dy}{dx}$ in terms of x and y.

(d) A curve is defined by the parametric equations $x(t)=t^2$ and $y(t)=\sin t$.
Calculate $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(e) Evaluate the following integrals.

(i) $\displaystyle\int\frac{7}{x^2}dx$

(ii) $\displaystyle\int\frac{1}{x-2}dx$

(iii) $\displaystyle\int x\cos x\ dx$

(iv) $\displaystyle\int-\sin\left(-\frac{x}{3}\right)dx$

(v) $\displaystyle\int_1^2 2x\sqrt{x^2-1}\ dx$

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TABLE OF INTEGRALS

[1] $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1$$

[2] $$\int \frac{dx}{x} = \ln|x| + c$$

[3] $$\int e^x\,dx = e^x + c$$

[4] $$\int \sin x\,dx = -\cos x + c$$

[5] $$\int \cos x\,dx = \sin x + c$$

[6] $$\int \tan x\,dx = \ln|\sec x| + c$$

[7] $$\int \sec^2 x\,dx = \tan x + c$$

[8] $$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$$

[9] $$\int \sinh x\,dx = \cosh x + c$$

[10] $$\int \cosh x\,dx = \sinh x + c$$

[11] $$\int \tanh x\,dx = \ln(\cosh x) + c$$

[12] $$\int x(ax+b)^n\,dx = \frac{1}{a^2}(ax+b)^{n+1}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + c, \quad n \neq -1, -2$$

[13] $$\int \frac{x^2}{ax+b}\,dx = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\ln|ax+b|\right] + c$$

[14] $$\int \frac{x^2}{(ax+b)^2}\,dx = \frac{1}{a^3}\left[ax+b - \frac{b^2}{ax+b} - 2b\ln|ax+b|\right] + c$$

[15] $$\int x\sqrt{ax+b}\,dx = \frac{2}{a^2}\left[\frac{(ax+b)^{5/2}}{5} - \frac{b(ax+b)^{3/2}}{3}\right] + c$$

[16] $$\int \frac{x}{\sqrt{ax+b}}\,dx = \frac{2ax-4b}{3a^2}\sqrt{ax+b} + c$$

[17] $$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c, \quad b > 0$$

[18] $$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

[19] $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

[20] $$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$$

[21] $$\int \frac{1}{(a^2-x^2)^2}\,dx = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right| + c$$

[22] $$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + c$$

[23] $$\int \frac{1}{(a^2-x^2)^{3/2}}\,dx = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + c$$

[24] $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + c$$

[25] $$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + c$$

[26] $$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right| + c$$

[27] $$\int \frac{1}{(x^2\pm a^2)^{3/2}}\,dx = \pm\frac{1}{a^2}\frac{x}{\sqrt{x^2\pm a^2}} + c$$

[28] $$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| + c$$

[29] $$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} - a\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right| + c$$

[30] $$\int \frac{1}{b+ke^{ax}}\,dx = \frac{1}{ab}\left[ax - \ln(b+ke^{ax})\right] + c, \quad ab \neq 0$$

[31] $$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + c$$

[32] $$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx) + c$$

[33] $$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

[34] $$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

[35] $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$$

[36] $$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

[37] $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

[38] $$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

[39] $$\int x^n \sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

[40] $$\int x^n \cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$$