2006 #1

Q4

(a i) Consider the function

$$y = \frac{-1}{B} \ln(1 + Ce^{-Dt})$$

Let $z = 1 + Ce^{-Dt}$ ($\frac{1}{2}$ mark)

$$y = \frac{-1}{B} \ln z \qquad z = 1 + Ce^{-Dt}$$

$$\frac{dz}{dt} = -CDe^{-Dt} \quad (1/2 \text{ mark})$$

$$\frac{dy}{dz} = \frac{-1}{B} \cdot \frac{1}{z} \quad (1/2 \text{ mark})$$

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt} \quad \text{[CHAIN RULE]} \quad (1 \text{ mark})$$

$$= \frac{-1}{B} \cdot \left(\frac{1}{1 + Ce^{-Dt}}\right)\left(-CDe^{-Dt}\right) \quad (1/2 \text{ mark})$$

$$= \frac{CD}{B} \cdot \frac{e^{-Dt}}{1 + Ce^{-Dt}}$$

3 marks

$$x = At - \frac{1}{B} \ln\left(1 + Ce^{-Dt}\right) + E$$

$$\frac{dx}{dt} = A + \left(\frac{CD}{B}\right) \cdot \frac{e^{-Dt}}{1 + Ce^{-Dt}}$$

1 mark

$\frac{dE}{dt} = 0$ 1 mark

Q4

(aii) $y = \cosh x \sinh x$

PRODUCT RULE $f = \cosh(x)$, $f' = \sinh(x)$

$g = \sinh(x)$, $g' = \cosh(x)$

$y' = gf' + fg' = \sinh(x).\sinh(x) + \cosh(x).\cosh(x)$

$\underline{y' = \sinh^2(x) + \cosh^2(x)}$ (2 marks) [Also $\cosh(2x)$]

(aiii) $y = \cot(x) = \dfrac{1}{\tan(x)}$

QUOTIENT RULE $f = 1$ $f' = 0$

1/2 mark

$g = \tan(x)$ $g' = \sec^2(x)$

1/2 mark 1/2 mark

$$\frac{dy}{dx} = \frac{gf' - fg'}{g^2} = \frac{0 - \sec^2(x)}{\tan^2(x)}$$

$$= \frac{-1}{\cos^2 x} \cdot \frac{1}{\tan^2 x} = \frac{-1}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x}$$

$= \underline{-\operatorname{cosec}^2 x}$ 1/2 mark

(2 marks)

#3

(b) $y^2 + y = \sin(x)$

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(y) = \frac{d}{dx}(\sin x)$$

$$\frac{d}{dy}(y^2)\cdot\frac{dy}{dx} + \frac{d}{dy}(y)\frac{dy}{dx} = \cos(x) \quad (1\text{ mark})$$

$$(2y+1)\frac{dy}{dx} = \cos(x)$$

$$\frac{dy}{dx} = \frac{\cos(x)}{1+2y} \qquad 2\text{ marks}$$

(c) $x = \sin(t^2) \qquad y = \cos(t)$

$$\frac{dx}{dt} = 2t\cos(t^2) \;\; {}^{1/2}\text{ mark} \qquad \frac{dy}{dt} = -\sin(t) \;\; {}^{1/2}\text{ mark}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{2t\cos(t^2)} \qquad 1\text{ mark}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-\sin(t)}{2t\cos(t^2)}\right)$$

$$z = 2t\cos(t^2)$$

$$= \frac{d}{dt}\left(\frac{-\sin(t)}{2t\cos(t^2)}\right)\cdot\frac{dt}{dx} \quad 1\text{ mark} \qquad \frac{dz}{dt} = 2\cos(t^2) + -4t\sin(t^2)$$

$$= -\left[\frac{2t\cos(t^2)\cdot\cos(t) - \sin(t)\left\{2\cos(t^2) - 4t\sin(t^2)\right\}}{4t^2\cos^2(t^2)}\right]$$

$$\cdot\frac{1}{2t\cos(t^2)} \qquad 1\text{ mark}$$

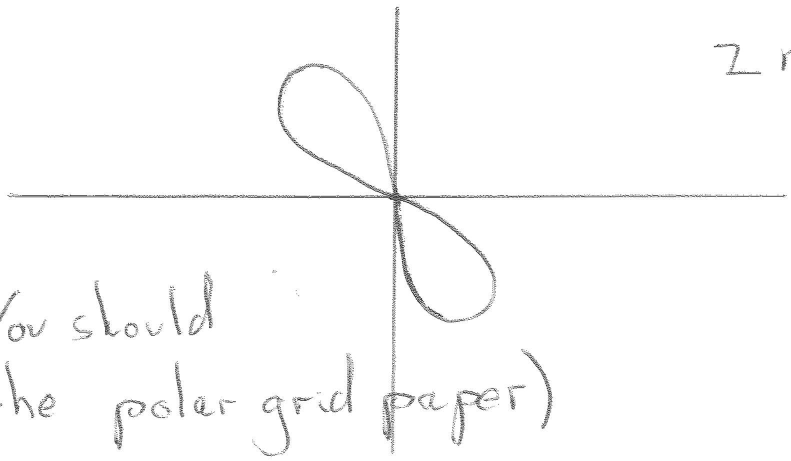
4 marks

(d) $r^2 = -8\sin\left[\frac{2\times 6.5\pi}{12}\right] = -8\sin\left[\frac{13\pi}{12}\right]$

$r^2 = 2.071$

$r = \pm 1.439$ 1 mark

(ii) 2 marks



(You should the polar grid paper)

e) i) $I = \int \left(\frac{12}{x} + x^5\right) dx = 12\ln(x) + \frac{x^6}{6} + c,$

where c is an arbitrary constant. 1 mark

ii) $I = \int \frac{dx}{x\sqrt{10-x^2}}$

Use Table 22 with $a^2 = 10$ 1/2 mark

$I = \frac{-1}{\sqrt{10}} \ln\left|\frac{\sqrt{10} + \sqrt{10 - x^2}}{x}\right| + C,$

where c is an arbitrary constant. 1 mark

Q4

(e iii)

$$I = \int_e^{3e} \sqrt{x^2+e^2}\, dx$$

Using Table 28 with $a^2 = e^2$ ½ mark

$$I = \left[\frac{1}{2} x\sqrt{x^2+e^2} + \frac{e^2}{2} \ln\left| x + \sqrt{x^2+e^2} \right| \right]_e^{3e}$$ ½ mark

$$= \frac{3e}{2}\sqrt{10e^2} + \frac{e^2}{2}\ln\left|3e + \sqrt{10e^2}\right|$$

$$- \frac{e}{2}\sqrt{2e^2} - \frac{e^2}{2}\ln\left|e + \sqrt{2e^2}\right|$$

$$= \frac{3\sqrt{10}}{2}e^2 + \frac{e^2}{2}\ln\left|3e + e\sqrt{10}\right| - \frac{\sqrt{2}}{2}e^2 - \frac{e^2}{2}\ln\left|e + e\sqrt{2}\right|$$

$$= \frac{1}{2}\left(3\sqrt{10} - \sqrt{2}\right)e^2 + \frac{e^2}{2}\ln\left|\frac{3+\sqrt{10}}{1+\sqrt{2}}\right|$$ 1 mark

(iv) Let $z = \ln(c)$

$$\frac{dz}{dc} = \frac{1}{c} \Rightarrow \frac{dc}{c} = dz$$

$$I = \int \frac{dc}{c \ln c} = \int \frac{dz}{z} = \ln z + d$$

$= \ln[\ln(c)] + d$, where d is an arbitrary constant (1 mark) ev) $I = \int \tan(x)\,dx$

$= \int \frac{\sin(x)}{\cos(x)}\,dx$

Let $v = \cos(x)$ 1/2

$\frac{dv}{dx} = -\sin(x) \Rightarrow \sin(x)\,dx = -dv$

$I = -\int \frac{dv}{v}$ 1/2 $= -\ln(v) + c$

$= \ln\left(\frac{1}{v}\right) + c = \ln\left|\frac{1}{\cos(x)}\right| + c$

$= \ln|\sec(x)| + c,$

where c is an arbitrary constant

[2marks]