MATH 111 — Applied Mathematical Modelling I

Spring Session 2006

Mid-Session Test

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Instructions

Time Allowed: 90 minutes
Number of questions: 8.

1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on both sides.
4. WORKING (including all necessary *reasoning*) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. At the beginning of each year you invest $q\%$ of you yearly salary into a superannuation scheme. Your employer tops up your investment by adding $r\%$ of your salary. Your money draws interest of $p\%$ compounded yearly.

   (a) Write down a **word** equation that defines you investment after n payments. [1]

$$\begin{bmatrix}\text{Money after}\\ n \text{ payments}\end{bmatrix} = \begin{bmatrix}\text{Money after}\\ (n-1) \text{ payments}\end{bmatrix} + \begin{bmatrix}\text{Interest earnt}\\ \text{on investments}\\ \text{after } (n-1)\\ \text{payments}\end{bmatrix} + \begin{bmatrix}\text{investment}\\ \text{after}\\ (n-1)\\ \text{payments}\end{bmatrix}$$

   (b) Suppose that your salary starts at a base level a and that at the start of every year it increases by a constant amount b.
   Write down, formally, the difference equation that describes your investment after n payments have been made. Define **all** variables and explain your terms. [2]

Your salary is $a + nb$

Your investment after $(n-1)$ payments is $\left[\frac{q+r}{100}\right](a+nb)$ [1/2]

Let Money after n payments be M_n [1/2]

Let interest rate be p

Then

$$M_n = M_{n-1} + \left(\frac{p}{100}\right)M_{n-1} + (a+nb)\left[\frac{q+r}{100}\right]$$

$$M_n = \left(1 + \frac{p}{100}\right)M_{n-1} + (a+nb)\left(\frac{q+r}{100}\right)$$ [1]

Don't solve the equation!

2. A debt of \$50,000 with interest at 11% compounded every two months is paid back by equal payments every two months over fifteen years.

   (a) What is the regular payment, rounded up to the nearest cent? [2]

   (b) How much profit does the lender make on the loan? [1]

(a) Use Loan Repayment Formula

$$D_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$

$D_0 = 50{,}000$ $p = 11$, $\alpha = 1/6$ $n = 6 \times 15$ [1]

$D_{90} = 0$. Let $\left(1 + \frac{\alpha p}{100}\right) = a$

$$0 = a^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$

$$\frac{100R}{\alpha p}\left(a^n - 1\right) = a^n D_0$$

$$R = \frac{\alpha p}{100} \cdot \frac{a^n D_0}{a^n - 1}$$ Note "rounded up"!

$$= \underline{\underline{\$1138.64}}$$ [1]

(b) Profit $= 90 \times 1138.64 - 50000$

$= \underline{\underline{\$52{,}477.60}}$ [1]

Note, "r

3. The number of chickens in Mr & Mrs Tweedy's farm is modelled by the difference equation,

$$c_n = (1 + g - \alpha)\, c_{n-1} - P, \quad n = 1, 2, 3 \ldots$$

where c_i is the number of chickens in the ith week, g is the fractional growth rate of chickens each week, α is the fraction of current chickens who are killed by foxes each week, and P is the constant number of chickens that are converted into pies each week. Assume that g and α are constant. For convenience let

$$\beta = 1 + g - \alpha,$$

and suppose that in week 0 there are c_0 chickens present.

(a) Find the general solution of the chicken model, simplifying as far as possible. [2]

(b) Suppose that $c_0 = 200$, $g = 0.25$ and $\alpha = 0.05$.

(i) Suppose that 20 chickens a week are converted into pies. What is the number of chickens on the farm in the limit $n \to \infty$? [2]

(ii) Suppose that 60 chickens a week are converted into pies. What is the number of chickens on the farm in the limit $n \to \infty$? [2]

(iii) What number of chickens (P) should be converted into pies each week if the number of chickens on the farm is to remain constant? [2]

(iv) At the beginning of the first week the pie machine breaks down before any chickens are converted into pies. It will take Mr. Tweedy eight weeks to fix the pie machine. When it is fixed Mrs Tweedy will convert all the chickens into pies. If one chicken produces four pies how many pies will Mrs Tweedy have? [2]

(a) $c_n = \beta^n c_0 + \sum_{p=1}^{n} \beta^{n-p}(-P)$ General formula for 1st order linear difference eq^n [$^1/_2$]

$= \beta^n c_0 - P \sum_{p=1}^{n} \beta^{n-p}$

Use formula for geometric progression [$^1/_2$]

$= \beta^n c_0 - P\left(\dfrac{\beta^n - 1}{\beta - 1}\right)$

$= \beta^n c_0 \quad \dfrac{-P\beta^n}{\beta - 1} \quad \dfrac{+P}{\beta - 1}$

$$c_n = \beta^n\left[c_0 - \frac{P}{\beta-1}\right] + \frac{P}{\beta - 1}$$ [1]

b) i) $c_0 = 200,\ g = 0.25,\ \alpha = 0.05$

$\beta = 1 + g - \alpha = 1 + 0.25 - 0.05 = 1.2$

$p = 20$

$$c_n = (1.2)^n \left[200 - \frac{20}{0.2} \right] + \frac{20}{0.2}$$

$$= (1.2)^n [100] + 100 \qquad [1]$$

Thus as $n \to \infty$ $\underline{\underline{\lim_{n\to\infty} c_n = +\infty}}$ [1]

(ii) $p = 60$ $\quad c_n = (1.2)^n \left[200 - \frac{60}{0.2} \right] + \frac{60}{0.2}$

$$c_n = 300 - 100(1.2)^n \qquad [1].$$

Thus $n \to \infty$ $c_n \to -\infty$. The number of chicken present is zero as they have all been converted into pies [1]

You don't get a mark for $c_\infty = -\infty$.

(iii) We need $c_0 - \frac{p}{\beta - 1} = 0$ [1] $\Rightarrow p = (\beta - 1)c_0$
$= 0.2(200)$
$\underline{\underline{p = 40}}$ [1]

(iv) Put $p = 0$ and $n = 8$

$c_n = c_0 \beta^n$

$= 200(1.2)^8$ or 859 (rounding down) [1]

NOT DIVIDED BY 4!

Pies = chickens $\times$ 4

$= 200(1.2)^8 \times 4$

$= 800(1.2)^8 = 3439.8 = 3439$ pies [1]

Don't round up. or 3436 pies (if rounded down)

(c) How do the chickens feel about being converted into pies? What should they do? [0]

4. The population of a species is modelled by the autonomous difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

Explain why it might be reasonable to replace this by a non-autonomous model

$$x_{n+1} = f(x_n, n), \quad n = 0, 1, 2, \ldots$$

The death-rate and/or birth rate might depend on the season / time of year [1]

5. Find the non-negative fixed points of a population governed by the model

$$x_{n+1} = \frac{3x_n^2}{x_n^2 + 2}, \qquad n = 0, 1, 2 \ldots$$

and check for stability. [4]

$$x = \frac{3x^2}{x^2+2}$$

½ for this if everything else wrong

$$x(x^2+2) - 3x^2 = 0 \quad [1]$$

$$x[x^2+2-3x] = 0$$

$$x[x-2][x-1] = 0$$

$$\Rightarrow \underline{x = 0, 1, 2} \quad [1]$$

$$f(x) = \frac{3x^2}{x^2+2}$$

$$f'(x) = \frac{(x^2+2)6x - (3x^2)2x}{(x^2+2)^2}$$

$$= \frac{12x}{(x^2+2)^2} \quad [½]$$

$\underline{x = 0}$

$\lambda = 0 \Rightarrow$ stable

$\underline{x = 1}$

$\lambda = \frac{12}{9} > 1 \Rightarrow$ unstable

$\underline{x = 2}$

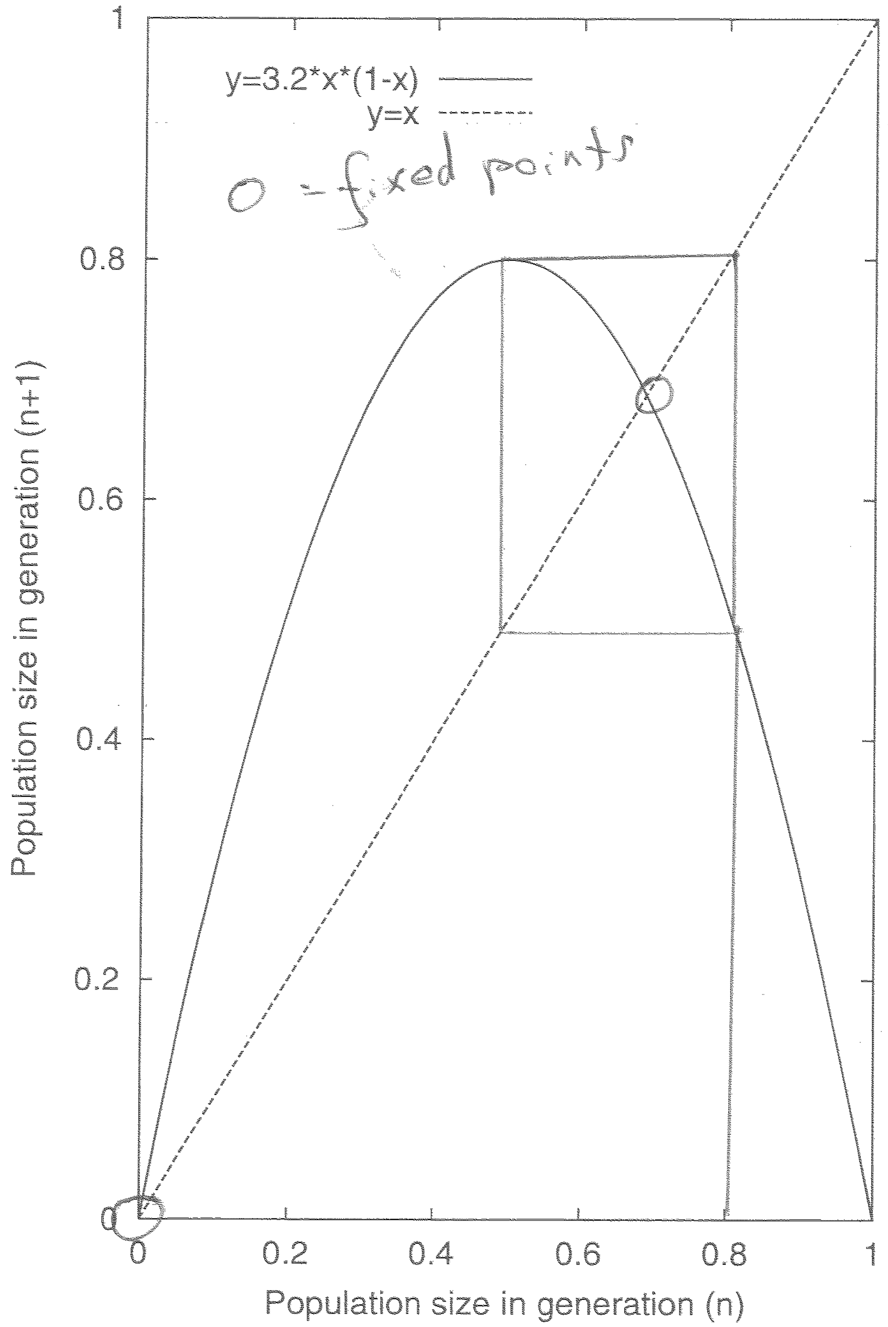
$\lambda = \frac{24}{36} < 1 \Rightarrow$ stable [½]

Note, fixed point stable if $-1 < \lambda < 1$

6. The population of a species is governed by the difference equation

$$x_{n+1} = 3.2x_n\,(1 - x_n)\,, \quad n = 0, 1, 2 \ldots.$$

The following figure shows the graphs $\dot{y} = 3.2x\,(1-x)$ and $y = x$.



(a) Find the fixed points of the model and determine their stability. [2]

(b) Mark the fixed points of the map on the figure. [1]

(c) Describe the algorithm used in cobwebbing. [2]

(d) Using the figure given, or otherwise, determine the long-time behaviour of the equation with the initial condition $x_0 = 0.8$. [2]

Can you observe?

(e) How does your answer to part (d) relate to the answer you gave in part (a)? [1]

(a) Fixed points

$x_{n+1} = f(x_n) \Rightarrow x = f(x)$

$x = 3.2x(1-x)$

$0 = x[3.2(1-x) - 1]$

$\underline{x = 0}$ ½ or $1 - x = \frac{1}{3.2}$ ½

$x = 1 - \frac{1}{3.2} = \frac{2.2}{3.2}$

Eigenvalue $\lambda = f'(x^*)$

$f = 3.2(x - x^2)$

$f' = 3.2(1-2x)$ [½]

$x = 0$ $\lambda = 3.2 \Rightarrow$ unstable [¼]

$x = \frac{2.2}{3.2}$ $\lambda = 3.2\left(1 - \frac{4.4}{3.2}\right)$

$= 3.2 - 4.4 = -1.2$

$\Rightarrow$ unstable. [¼]

Note stable if $-1 < \lambda < 1$.

(c) Draw a vertical line from the value x_0 to the curve $y = f(x)$. Then draw a horizontal line to the 'curve' $y = x$. Then draw a vertical line to the curve $y = f(x)$. Then draw a horizontal line to the curve $y = x$ etc

(d) The cob-webbing diagram suggests a period-2 solution.

(e) Certainly the cob-webbing diagram can't show that the solution converges to either of the fixed points because we have shown that these are both unstable.

7. (a) Show that by defining

$$x_t \equiv \frac{1}{rK} \cdot N_t$$

the logistic difference equation with constant harvesting

(1) $$N_{t+1} = \left(r - \frac{N_t}{K}\right) N_t - H$$

can be transformed into its standard form

(2) $$x_{t+1} = r(1 - x_t)\, x_t - h,$$

where $h = \frac{H}{rK}$. In equation (1) H is the number of animals killed each year, K is the carrying capacity, and r is the static birth rate. [2]

(b) A population of sandhill cranes (*Grus canadiensis*) has been modelled by a logistic equation with carrying capacity of 194,600 members and intrinsic growth rate 1.0987year^{-1}.

(i) Find, to four decimal places, the critical harvest rate (h_{cr}) for which constant yield harvesting will drive the population to extinction. Find the corresponding, biologically meaningful, value of H_{cr}. [2]

(ii)
- Suppose that 300 birds a year are killed. What is the corresponding value for h? (state your answer to four decimal places). [1]
- Using your answer to the previous part of this question what is the equilibrium population size of equation (2) under constant yield harvesting of 300 birds per year? [2]
- Hence, or otherwise, find the equilibrium population size of equation (2) under constant yield harvesting of 300 birds per year. [2]

You may quote appropriate results from your lecture notes.

(a)
$$\left.\begin{aligned} N_t &= rK x_t \\ N_{t+1} &= rK x_{t+1} \end{aligned}\right\} [1]$$

$$rK x_{t+1} = \left(r - \frac{rK x_t}{K}\right) rK x_t$$

$$\underline{x_{t+1} = r(1 - x_t)x_t} \qquad [1]$$

(b)
$$h_{cr} = \frac{(r-1)^2}{4r} = \frac{(1.0987-1)^2}{4 \times 1.0987}$$

$$= 0.0022 \qquad 2.2166 \times 10^{-3} \qquad [1]$$

$$H_{cr} = rKh_{cr}$$
$$= 1.0987(194600)(0.0022)$$
$$= 470.37 = \underline{471} \qquad [1]$$

b ii) • $h = \dfrac{H}{rK} = \dfrac{300}{1.0907(194600)}$

$= \underline{\underline{0.0014}}$ [1]

• The stable fixed point is

$x^* = \dfrac{-(1-r) + \sqrt{(1-r)^2 - 4rh}}{2r}$ [1]

$= \underline{\underline{0.0722}}$ [1]

[1]

• $N^* = rKx^*$

$= 15436.8$

$= \underline{\underline{15436}}$ or $\underline{\underline{15437}}$. [1]

8. A chemical reactor is being used to grow micro-organisms. The productivity of the reactor is given by

$$P = \frac{St - (1 + S)}{(t - 1)\,t},$$

where S is the substrate concentration flowing into the reactor and t is the residence time.

Write appropriate maple code to plot the productivity as a function of the residence time when $S = 1$ over the range of values $2 \leq t \leq 20$. [2]

```
f := (S*t -(1+S))/((t-1)*t);
S := 1;                           [1]
plot(f, t=2..20);                 [1]
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