MATH 111 — Applied Mathematical Modelling I

Spring Session 2007

Mid-Session Test

Student Name: ______________________ *Student Number*: ____________

Instructions

Time Allowed: 90 minutes
Number of questions: 7.

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1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on both sides.
4. WORKING (including all necessary *reasoning*) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. Consider the difference equation

$$x_{n+2} - 3x_{n+1} + 2x_n = 0.$$

(a) State the order of this equation, whether it is linear or nonlinear and whether it is autonomous or non-autonomous. You must justify your answers. [1]

(b) Show that the general solution to the equation is given by

$$x_n = A_1 + 2^n A_2,$$

where A_1 and A_2 are constants. [3]

(c) Using the general solution, solve for the specific solution with initial conditions $x_0 = 10$ and $x_1 = 20$. [2]

a) Order is $(n+2) - (n) = 2$
Equation is linear - each term is linear.
Equation is autonomous as the time variable 'n' only appears as a subscript (only right, then ½ mark) [1]

b)
$$\left.\begin{aligned} x_n &= A_1 + 2^n A_2 \\ x_{n+1} &= A_1 + 2^{n+1} A_2 \\ x_{n+2} &= A_1 + 2^{n+2} A_2 \end{aligned}\right\} 1$$

$$x_{n+2} - 3x_{n+1} + 2x_n = A_1 + 2^{n+2}A_2 - 3A_1 - 3A_2 . 2^{n+1} + 2A_1 + 2 . 2^n A_2$$

$$= A_2\left[2^{n+2} - 3.2^{n+1} + 2.2^n\right] \quad 1$$

$$= A_2 . 2^n\left[2^2 - 3.2 + 2\right] = 0 \quad 1$$

c)
$$\left.\begin{aligned} x_0 &= 10 = A_1 + A_2 \quad (*) \\ x_1 &= 20 = A_1 + 2A_2 \quad (**) \end{aligned}\right\} 1$$

$(**) - (*) \Rightarrow A_2 = 10$

$\Rightarrow A_1 = 0$ $\Big\}$ 1/2

$\therefore\ X_n = 10 \cdot 2^n$

2. A bird species lives on two islands, North Island and South Island. On the North Island the population increases by 25% each year whereas on the South Island it increases by 10% each year. Each year 400 birds leave the North Island, one-quarter of which go to the South Island.

   (a) Write down a **word** equation that defines this problem. [1]

Change in bird Numbers on North Island = population increase − migration

Change in bird numbers on South Island = population increase + immigration

   (b) Write down, formally, the set of difference equations that models this problem after n years. Define **all** variables and explain your terms. [2]

Let N_N and S_N be the number of birds on the North and South island respectively in year n.

$$N_n - N_{n-1} = 0.25 N_{N-1} - 400$$

$$N_n = 1.25 N_{n-1} - 400$$

$$S_N - S_{n-1} = 0.10 S_{N-1} + 100$$

$$S_N = 1.1 S_{N-1} + 100$$

3. At the beginning of each year you invest $q\%$ of your yearly salary into a superannuation scheme. Your employer tops up your investment by adding $r\%$ of your salary. Your money draws interest of $p\%$ compounded yearly. Suppose that your salary starts at a base level b and at the start of each year it increases by a constant amount c. Your investment in the superannuation scheme after n payments is given by the solution of the difference equation

$$M_n = \left(1 + \frac{p}{100}\right) M_{n-1} + \left(\frac{q+r}{100}\right)(b + cn), \quad M_0 = \frac{q+r}{100} b.$$

(a) Find the general solution of the superannuation model, simplifying as far as possible. [5]

Hint. $\displaystyle\sum_{k=1}^{n} a^{n-k} k = \frac{a^{n+1} - (n+1)a + n}{(a-1)^2}.$

(b) Suppose that $q = 7$, $r = 14$, $b = \$50,000$, $c = \$1000$ and the rate of return from the superannuation scheme is 4%.

(i) What is the size of your initial investment? Round your answer to the nearest cent. [1]

(ii) You retire from work after making 35 payments. How much money do you have in your superannuation scheme? [2]

(c) How much money would you have in your superannuation scheme when you retired if you did not increase your contribution in line with your increasing salary, i.e. your regular investment is fixed at its initial value and $c = 0$. Assume that the values for b, p, q & r are as in part (b). [2]

Let $\alpha = 1 + \frac{p}{100}$, $\beta = \left(\frac{q+r}{100}\right)$

$$M_n = \alpha M_{n-1} + b\beta + c\beta n$$

General sol^n to

$$x_n = a x_{n-1} + b(n) \text{ is}$$

$$x_n = a^n (x_0) + \sum_{p=1}^{n} a^{n-p} b(p) \quad 1$$

$$M_n = \alpha^n M_0 + \sum_{p=1}^{n} \alpha^{n-p}\left[b\beta + c\beta p\right] \quad 1/2$$

$$= \alpha^n M_0 + b\beta \sum_{p=1}^{n} \alpha^{n-p} + c\beta \sum_{p=1}^{n} p\alpha^{n-p} \quad 1/2$$

Now 4b)

$$\sum_{p=1}^{n} \alpha^{n-p} = \frac{\alpha^n - 1}{\alpha - 1}$$ as it is a Geometric Progression. 1

$$\sum_{p=1}^{n} p\alpha^{n-p} = \frac{\alpha^{n+1} - (n+1)\alpha + n}{(\alpha-1)^2}$$ using hint. 1

So

$$M_n = \alpha^n M_0 + b\beta\left[\frac{\alpha^n - 1}{\alpha - 1}\right] + c\beta\left\{\frac{\alpha^{n+1} - (n+1)\alpha + n}{(\alpha-1)^2}\right\}.$$ 1

b i) $M_0 = \frac{q+r}{100} b = \frac{21}{100} \cdot 50{,}000 = 10{,}500$

ii) $\alpha = 1 + \frac{p}{100} = 1 + \frac{4}{100} = 1.04$

$\beta = \frac{q+r}{100} = \frac{21}{100}$

$$M_{35} = (1.04)^{35}(10{,}500) + 50{,}000 \times \frac{21}{100} \times \left[\frac{1.04^{35} - 1}{1.04 - 1}\right]$$

$$+ 1000 \times \frac{21}{100}\left\{\frac{(1.04)^{36} - (36)(1.04) + 35}{(1.04-1)^2}\right\}$$

$$\underline{\underline{M_{35} = 1.033 \times 10^6}}$$

(c) $c = 0$

$$M_n = \alpha^n M_0 + b\beta\left[\frac{\alpha^n - 1}{\alpha - 1}\right]$$

$$= (1.04)^{35}(10{,}500) + \frac{21}{100}\, 50{,}000\left[\frac{1.04^{35} - 1}{1.04 - 1}\right]$$

$$= \underline{\underline{814{,}782.30}}$$

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4. On the 18th August 2007 I visited a Bing Lee store. I noticed that the following terms were offered for a 48-month loan.

   - Interest is compounded at 17.9% monthly.
   - There is a $25 establishment fee.
   - At the end of each month there is a $2.95 monthly service fee.

   Suppose that you purchase an item at Bing Lee costing $1,000 and that you take a loan for 100% of this amount. How much money do you repay to Bing Lee? [3]

Loan Repayment Formula

$$0 = \left[1 + \frac{\alpha p}{100}\right]^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$

Let $\frac{\alpha p}{100} = i$

$$0 = (1+i)^n \left(D_0 - \frac{R}{i}\right) + \frac{R}{i}$$

$$\frac{(1+i)^n}{i} R - \frac{R}{i} = (1+i)^n D_0$$

$$\frac{R}{i}\left[(1+i)^n - 1\right] = (1+i)^n D_0$$

$$R = \frac{(1+i)^n D_0 \, i}{[(1+i)^n - 1]}$$

$$i = \frac{(1/12)(17.9)}{100}, \quad D_0 = 1{,}000, \quad n = 48$$

$$R = 29.33 \quad [1]$$

Payments to Bing Lee are

$48 \times 29.33 + 25 + 48 \times 2.95$ [1]

$= \$\ \underline{\underline{1574.44}}$

5. A population is modelled by the difference equation

$$x_{n+1} = f(x_n), n = 0, 1, 2, \ldots$$

where the function $f(x)$ has the following properties.

- $f(0) = 0$.
- If $x > 0$ then $f(x) > 0$.
- If $x > 0$ then $f(x) < x$.

(a) What properties must the function f have in order for it to be biologically meaningful? [2]

(b) Produce a possible cob-webbing diagram by sketching the function $y = f(x)$ and drawing the line $y = x$ [1]

(c) By using your cob-webbing diagram, or otherwise, determine the long-term population dynamics for an arbitrary initial condition $x_0 = a > 0$. [2]

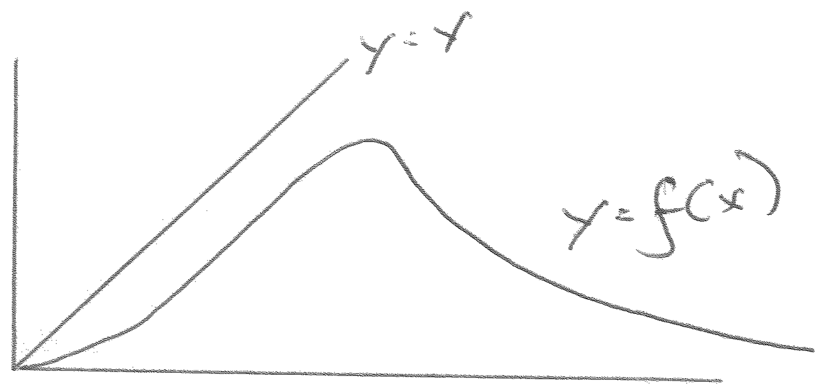
(a) i) $f(0) = 0$

ii) If $x > 0$ $f(x) > 0$

iii) $\lim_{x \to \infty} f(x) = 0$

iv) $f(x)$ only has 1 local maximum on $[0, \infty)$.

(b)



(c) For any initial condition $x_0 = a > 0$

$\lim_{n \to \infty} x_n = 0$ i.e. the population

becomes extinct.

6. The standard logistic model is given by

$$x_{n+1} = r x_n (1 - x_n),$$

where r is the intrinsic growth rate, a positive number. If a proportion, p, of the population is culled before natural births and deaths occur we obtain a modified logistic model

$$x_{n+1} = r(1-p)x_n [1 - (1-p)x_n].$$

(a) Find the fixed points of the modified logistic model. [3]

(b) Determine the eigenvalue of the trivial fixed point. [2]

(c) The value for r for a pest in a specific environment is $r = 2$. Will a value $p = 0.4$ suffice to drive the species to distinction? (Justify your answer). If not, suggest a value for p that will drive the species to extinction. (Justify your value) [4]

a) $x = r(1-p)x[1-(1-p)x]$

$0 = x\{r(1-p)[1-(1-p)x] - 1\}$

$\underline{x = 0}$ or $r(1-p)[1-(1-p)x] = 1$

$1-(1-p)x = \frac{1}{r(1-p)}$

$1 - \frac{1}{r(1-p)} = (1-p)x$

$$\underline{x = \frac{r(1-p)-1}{r(1-p)^2}}$$

b) $\lambda = f'(x)$

$f(x) = r(1-p)[x - (1-p)x^2]$

$f'(x) = r(1-p)[1 - 2(1-p)x]$

$\underline{f'(0) = r(1-p)}$

(c) We need $-1 < \lambda < 1$ for the trivial fixed point to be stable.

$R = 2, \ p = 0.4$

$\lambda = 2(1-0.4) = \frac{2 \cdot 6}{10} = \frac{12}{10} > 1.$

Steady state is unstable - population not driven to extinction.

Need
$$r(1-p) < 1$$
$$2(1-p) < 1$$
$$1-p < 1/2$$
$$1/2 < p.$$

Any value for p with $p > 1/2$ will do.

7. In order to draw a cobwebbing diagram for the population model

$$x_{n+1} = \frac{2x_n^2}{x_n^2 + 3}$$

A figure containing the straight line $y = x$ and the curve $y = \dfrac{2x^2}{x^2+3}$ is required for a cobwebbing diagram.

(a) Why is the straight line $y = x$ required in a cobwebbing diagram? [1]

(b) Write appropriate maple code to plot the straight line $y = x$ and the curve $y = \dfrac{2x^2}{x^2+3}$ on the same figure over the range $0 \leq x \leq 10$. [3]

a) In a cobwebbing diagram you draw a vertical line from x to f(x) and then a horizontal line to y=x to obtain the point [f(x), f(x)].

b)
```
g := x ;
f := 2*x^2/(x^2+3);
plot({f,g}, x=0..10);
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