University of Wollongong

School of Mathematics and Applied Statistics

MATH 111 — Applied Mathematical Modelling I (Wollongong Campus)

Spring Session Examination 2006~~6~~7 !

Family Name	NELSON
First Name	MARK
Student Number	
Table Number	

Time Allowed: 2 hours and 15 minutes

Number of Questions: 8.

Directions to Candidates

1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on one side only.
4. WORKING (including all necessary reasoning) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. A group of individuals who have a particular disease are involved in a drug trial. At the beginning of each week a fraction β of those who still have the disease are given a drug. The drug successfully cures a fraction f of those treated.

   (a) Write down a **word** equation that describes this problem. [1]

   (b) Write down, formally, the difference equation that models the number of infected individuals after n weeks of the drug trial. Define **all** variables and explain your terms. [2]

   (c) Find the general solution of your model. [1]

   (d) At the start of the drug trial there are 100 individuals with the disease. Each week 10% of the infected individuals are treated. After four weeks 25 individuals have been cured. Estimate the value for f. [3]

a) Change in Number of infected individuals $= -$[Number cured]

b) Let I_n be the number of infected individuals after n weeks. Then

$$(I_n - I_{n-1}) = -\beta f I_{n-1}$$

$$\underline{\underline{I_n = (1-\beta f) I_{n-1}}}$$

Note β, f are defined in the question.

1 mark for eq^n, 1 mark for definition

c) $\underline{\underline{I_n = (1-\beta f)^n I_0}}$

If (a) is wrong but (b) is right then maximum mark is $1\frac{1}{2}$

d) We given $I_0 = 100$

$$I_4 = 100 - 25 = 75$$

$$f = 0.1$$

$$I_n = (1-\beta f)^4 I_0$$

$$75 = (1-0.1\beta)^4\, 100 \qquad \text{[1 mark]}$$

$$\left(\frac{3}{4}\right)^{1/4} = 1 - 0.1\beta$$

$$0.1\beta = 1 - \left(\frac{3}{4}\right)^{1/4}$$

$$\beta = 10\left[1 - \left(\frac{3}{4}\right)^{1/4}\right] \cong 69.4\%$$

[0.694]

If answer to (c) is wrong BUT answer to (d) is correct then 2 marks

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2. The number of birds on an island at the end of each year is modelled by the equation

$$N_t = (1+g)\,N_{t-1} - i, \quad t = 0, 1, 2\ldots$$

where g is the net growth rate of the birds, i is the number of birds who migrate each year and t is the time in years.

(a) Find the general solution of the bird model, simplifying as far as possible. [2]

(b) Assume that $N_0 = 800$ and $g = 0.25$.

(i) Suppose that 100 birds a year leave the island. What is the number of birds on the island in the limit as $n \to \infty$? [2]

(ii) Suppose that 400 birds a year leave the island. What is the number of birds on the island in the limit as $n \to \infty$? [2]

(iii) What number of birds should migrate each year to maintain a constant population size on the island? [2]

a) Let $a = 1+g$.
The equation $N_t = aN_{t-1} - i$ has general solution

$$N_t = a^t N_0 + \sum_{p=1}^{t} a^{n-p}[(-i)]$$

$$= N_0 a^t - i \sum_{p=1}^{t} a^{n-p}$$

$$= N_0 a^t - \frac{i[a^n - 1]}{a-1} = N_0 a^t - \frac{i[(1+g)^t - 1]}{(1+g)-1}$$

$$= N_0 (1+g)^t - \frac{i[(1+g)^t - 1]}{g}$$

b) It is useful to rewrite the answer to (a) in the form

$$N_t = \frac{i}{g} + \left[N_0 - \frac{i}{g}\right](1+g)^t$$

(i) $N_0 = 800 \quad g = 0.25 \quad i = 100$

$$N_t = \frac{100}{0.25} + \left(800 - \frac{100}{0.25}\right)\left(1.25\right)^t$$

$$= 400 + 400(1.25)^t.$$

$$\therefore \lim_{t \to \infty} N_t = +\infty$$

(ii) $N_0 = 800 \quad g = 0.25 \quad i = 400$

$$N_t = \frac{400}{0.25} + \left(800 - \frac{400}{0.25}\right)\left(1.25\right)^n$$

$$= 1600 - 800(1.25)^n$$

$\therefore \lim_{t \to \infty} N_t = -\infty$ mathematically, but practically

$$\lim_{t \to \infty} N_t = 0$$

[1½ marks for "$-\infty$"]
[1½ marks for saying 'extinct!']

2) Need $\left[N_0 - \frac{i}{g}\right] = 0 \Rightarrow i = N_0 g$

$$i = 200$$

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3. Consider the difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2 \ldots$$

Explain in terms understandable to a Year-12 student studying mathematics:

(a) What *fixed points* are and how they may be found;

(b) How finding the fixed points provides important insights into the long-term dynamics of the difference equation.

[4]

a) Fixed points are values of x which have the property that $x_{n+1} = x_n$

i.e $x_{n+1} = f(x_n) = x_n$

They can be found by either

(i) solving the equation $x = f(x)$

(ii) finding the points of intersection of the curves $y = x$ and $y = f(x)$

b) Fixed points represent equilibrium points of the system, because $x_{n+1} = x_n$ the value does not change with time. This is the simplest possible behaviour a system can exhibit.

4. A drug delivery vehicle, such as a skin patch, is attached to a patient. The drug moves from the patch into the skin, and from the skin into the blood stream. The amount of drug per unit area remaining in the delivery device (A) and the amount of drug per unit area in the skin (X) are described by the pair of differential equations

$$(1) \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = -k_{21}A, \qquad A(0) = A_0,$$

$$(2) \qquad \frac{\mathrm{d}X}{\mathrm{d}t} = -k_{1B}X + k_{21}A, \quad X(0) = 0.$$

In these equations A_0 is the initial concentration of drug per unit area in the delivery device. The parameter k_{21} is the rate at which drug enters the skin from the vehicle. The parameter k_{1B} is the rate at which drug leaves the skin and enters the blood stream.

(a) Solve the system of differential equations to find the amount of drug per unit area in the vehicle and skin as a function of time. [7]

(Hint. First solve differential equation (1) and then substitute your answer into differential equation (2). Then solve differential equation (2). You may assume that $k_{1B} \neq k_{21}$).

(b) Find the value of time, t_{max}, at which the concentration of drug in the skin (X) reaches its maximum value. [3]

(c) The cumulative amount of drug per unit area excreted from the skin into the bloodstream (Ae_s) is expressed by

$$Ae_s(t) = k_{1B}\int_0^t X\mathrm{d}t, \quad Ae_s(0) = 0.$$

Determine the cumulative amount of drug per unit area excreted from the skin into the blood stream in the limit that $t \to \infty$, i.e.

$$Ae_s(\infty) = k_{1B}\int_0^\infty X\mathrm{d}t.$$

[3]

(d) What is the significance of your answer to part (4c) of this question? [1]

a) $\dfrac{dA}{dt} = -k_{21}A \quad , \quad A(0) = A_0$

$A = A_0 \exp[-k_{21}t]$. [2 marks]

$$\frac{dX}{dt} + k_{1B}X = k_{21}A_0 \exp[-k_{21}t] \qquad [1/5]$$

$$IF = \exp\int k_{1B}\,dt = \exp[k_{1B}t] \qquad [2/5]$$

$$\frac{d}{dt} X\exp[k_{1B}t] = k_{21}A_0 \exp[(k_{1B}-k_{21})t] \qquad [3/5]$$

$$X\exp[k_{1B}t] = \frac{k_{21}A_0 \exp[(k_{1B}-k_{21})t]}{k_{1B}-k_{21}} + A, \qquad [4/5]$$

where A is a constant of integration.

Initial condition $X(0)=0 \Rightarrow A = \dfrac{-k_{21}A_0}{k_{1B}-k_{21}}$ [5/5]

$$\therefore X(t) = \frac{k_{21}A_0}{k_{1B}-k_{21}}\left[\exp(-k_{21}t) - \exp(-k_{1B}t)\right] \qquad [5 \text{ marks}]$$

b) The maximum value of X occurs when $\dfrac{dX}{dt}=0$ [1 mark]

$$\frac{dX}{dt} = \frac{k_{21}A_0}{k_{1B}-k_{21}}\left[\begin{matrix}-k_{21}\exp(-k_{21}t) \\ +k_{1B}\exp(-k_{1B}t)\end{matrix}\right]$$

[1 mark]

$\dfrac{dX}{dt}=0 \Rightarrow$

$$k_{1B}\exp(-k_{1B}t) = k_{21}\exp(-k_{21}t)$$

$$\exp[(k_{21}-k_{1B})t] = \frac{k_{21}}{k_{1B}}$$

$$(k_{21}-k_{1B})t = \ln\left(\frac{k_{21}}{k_{1B}}\right)$$

[1 mark]

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$$t_{max} = \frac{1}{k_{21} - k_{1B}} \ln\left(\frac{k_{21}}{k_{1B}}\right)$$

c) $$A_{e_S}(\infty) = k_{1B} \int_0^\infty \frac{k_{21} A_0}{k_{1B} - k_{21}} \left[\exp(-k_{21}t) - \exp(-k_{1B}t)\right] dt$$

$$= \frac{k_{1B} k_{21} A_0}{k_{1B} - k_{21}} \left[\frac{-\exp(-k_{21}t)}{k_{21}} + \frac{\exp(-k_{1B}t)}{k_{1B}}\right]_0^\infty$$

$$= \frac{k_{1B} k_{21} A_0}{k_{1B} - k_{21}} \left[0 - \left(\frac{-1}{k_{21}} \ \frac{+1}{k_{1B}}\right)\right] = \frac{k_{1B} k_{21} A_0}{k_{1B} - k_{21}} \left(\frac{1}{k_{21}} \ \frac{-1}{k_{1B}}\right)$$

$$= \frac{k_{1B} k_{21} A_0}{k_{1B} - k_{21}} \left(\frac{k_{1B} - k_{21}}{k_{1B} k_{21}}\right) = A_0$$

d) The significance is that all the drugs delivered into the blood stream as $t \to \infty$.

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5. The population of fish is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

where the function $f(x)$ is given by

$$f(x) = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K_0} - 1\right),$$

where $r > 0$ and $0 < K_0 < K$.

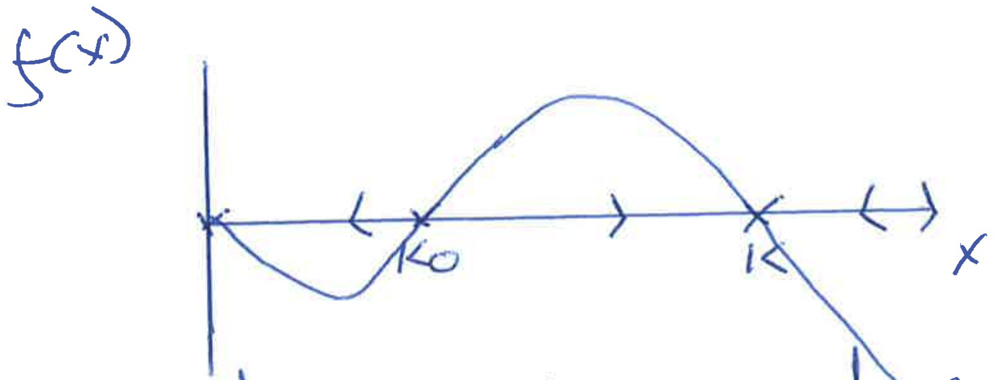
(a) Sketch the growth curve $f(x)$ as a function of x. [2]

(b) Using your sketch determine the stability of the steady-state solutions $x = 0$, $x = K_0$ and $x = K$, carefully explaining your reasoning. [3]

(c) How does the long-term evolution of the differential equation depend upon the choice of the initial condition x_0? [3]

(d) A disease spreads through the population reducing the population density to $K_0/2$. What happens to the population? Justify your answer. [2]

a)



b) The steady-states at $x=0$ and $x=K$ are stable. The steady-state at $x=K_0$ is unstable. To see this consider the direction of the derivites near the steady-state solutions

[half marks for a correct description of a wrong figure] c)

If $x_0 \in (0, K_0)$ then $\lim_{t \to \infty} x(t) = 0$

If $x_0 \in (K_0, K) \cup (K, \infty)$ then $\lim_{t \to \infty} x(t) = K$

$1\frac{1}{2}$ marks

If $x_0 = 0$ then $x(t) = 0 \ \forall t$.

If $x_0 = K_0$ then $x(t) = K_0 \ \forall t$

If $x_0 = K$ then $x(t) = K \ \forall t$

$1\frac{1}{2}$ mark

[Half marks for a correct description of a wrong figure]

d) As $\frac{K_0}{2} < K$ it follows from part (c) that the population becomes extinct

[Half marks for a correct answer for a wrong figure]

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6. The population density of the spruce budworm in a forest is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{q}\right) - \frac{x^2}{1+x^2}.$$

Determine the stability of the trivial steady-state solution $x = 0$. [2]

$$f = r\left[x - \frac{x^2}{q}\right] \quad \frac{-x^2}{1+x^2}$$

$$f'(x) = r\left(1 - \frac{2x}{q}\right) - \left[\frac{2(1+x^2)x - x^2(2x)}{(1+x^2)^2}\right]$$ [1]

$$f'(0) = r > 0.$$

$\therefore$ $x=0$ is unstable. [1]

[We can assume $r > 0$, as it is a growth rate]

[If eigenvalue calculation is wrong, by which I mean $\frac{df}{dx}$ is incorrect but you correctly substituted $x=0$ into your wrong equation then 1/2 a mark]

7. Consider the logistic equation with constant effort harvesting

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right) - Ex, \quad x(0) = K,$$

where x is the population density of an animal in an environment, t time, $r > 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity of the environment and $E \geq 0$ is the effort expanded in harvesting.

(a) Determine the steady-state solutions of the logistic equation with constant effort harvesting. [3]

(b) Suppose that the value for the intrinsic growth rate is $r = 2$.

(i) Determine the stability of the steady-state solutions as a function of the effort E. [3]

(ii) Draw a steady-state diagram for the logistic equation with constant effort harvesting for the case $r = 2$ showing how the steady-state solutions of the model vary as a function of the effort expanded in harvesting. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively. [3]

(iii) For what value of the parameter E does a bifurcation occur? What kind of bifurcation is it? [2]

Steady-states are when $\frac{dx}{dt} = 0$

a) $0 = rx\left(1 - \frac{x}{K}\right) - Ex$

$0 = x\left[r\left(1 - \frac{x}{K}\right) - E\right]$ $\qquad \underline{\underline{x = 0}}$

or $r\left(1 - \frac{x}{K}\right) - E = 0$

$1 - \frac{x}{K} = \frac{E}{r}$

$1 - \frac{E}{r} = \frac{x}{K}$ $\qquad x = K\left(\frac{r-E}{r}\right)$

b i) $\lambda = f'(x) = \frac{d}{dx}\left[r\left(x - \frac{x^2}{K}\right) - Ex\right] = r\left(1 - \frac{2x}{K}\right) - E$

$r = 2,\ x = 0$

$\lambda = 2 - E$

Stable $2 - E < 0 \Rightarrow E > 2$

Unstable $2 - E > 0 \Rightarrow E < 2$

$r = 2,\ x = K\left(\frac{2-E}{2}\right)$

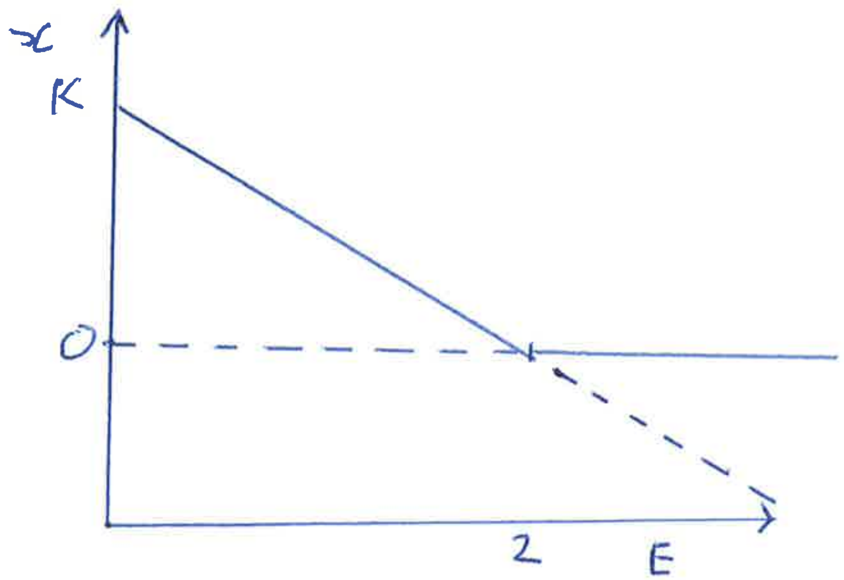
$\lambda = 2\left[1 - \frac{\cancel{2}}{K} \cdot \frac{K}{\cancel{2}}[2-E]\right] - E$

$= 2[E-1] - E \quad = E - 2$

stable $E - 2 < 0 \Rightarrow E < 2$

unstable $E - 2 > 0 \Rightarrow E > 2$

ii)



iii) A transcritical bifurcation occurs at $E = 2$.

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8. Thermal and solutal dispersion in a circular tube with diffusion into the wall is characterised by a dispersion coefficient (Λ) which is a function of the void fraction of the bed (ϵ_f) and the ratio of fluid thermal diffusivity to diffusivity in the wall (μ).

The function Λ is defined by the following maple code

```
L  := g1/48.0 +mu*g2/8.0;
g1 := epsilon*(6*epsilon^2 -16*epsilon+11);
g2 := epsilon*(4*epsilon -epsilon^2 -3 -2*ln(epsilon));
```

where the function Λ is defined as L, the parameter ϵ_f is defined as epsilon and the parameter μ is defined as mu. In the following, you do *not* need to write out the definitions of the functions L, g1 and g2.

Write maple code to find the value of epsilon that maximises the function L, and the corresponding maximum value of the function L, when $\mu = 0.1$. [4]

```
mv    = 0.1;
Le    := diff(L, epsilon);
epsilon := fsolve(Le, epsilon);
L;
```

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