

University of Wollongong  
School of Mathematics and Applied Statistics  
**MATH 111 — Applied Mathematical  
Modelling I (Wollongong Campus)**  
Spring Session Examination 2007

Family Name	_____
First Name	_____
Student Number	_____
Table Number	_____

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**Time Allowed:** 2 hours and 15 minutes

Number of Questions: 8.

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**Directions to Candidates**

1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on one side only.
4. WORKING (including all necessary reasoning) is to be shown for all solutions.
5. Working is to be done in the exam paper.

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**Examination Materials/Aids Allowed**

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

**Examination Materials/Aids to be supplied**

None.

**This examination paper must NOT be  
removed from the examination room.**

1. A group of individuals who have a particular disease are involved in a drug trial. At the beginning of each week a fraction  $\beta$  of those who still have the disease are given a drug. The drug successfully cures a fraction  $f$  of those treated.
  - (a) Write down a **word** equation that describes this problem. [1]
  - (b) Write down, formally, the difference equation that models the number of infected individuals after  $n$  weeks of the drug trial. Define **all** variables and explain your terms. [2]
  - (c) Find the general solution of your model. [1]
  - (d) At the start of the drug trial there are 100 individuals with the disease. Each week 10% of the infected individuals are treated. After four weeks 25 individuals have been cured. Estimate the value for  $f$ . [3]

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2. The number of birds on an island at the end of each year is modelled by the equation

$$N_t = (1 + g) N_{t-1} - i, \quad t = 0, 1, 2 \dots$$

where  $g$  is the net growth rate of the birds,  $i$  is the number of birds who migrate each year and  $t$  is the time in years.

- (a) Find the general solution of the bird model, simplifying as far as possible. [2]
- (b) Assume that  $N_0 = 800$  and  $g = 0.25$ .
- (i) Suppose that 100 birds a year leave the island. What is the number of birds on the island in the limit as  $t \rightarrow \infty$ ? [2]
- (ii) Suppose that 400 birds a year leave the island. What is the number of birds on the island in the limit as  $t \rightarrow \infty$ ? [2]
- (iii) What number of birds should migrate each year to maintain a constant population size on the island? [2]

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3. Consider the difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2 \dots$$

Explain in terms understandable to a Year-12 student studying mathematics:

- (a) What *fixed points* are and how they may be found;
- (b) How finding the fixed points provides important insights into the long-term dynamics of the difference equation.

[4]

4. A drug delivery vehicle, such as a skin patch, is attached to a patient. The drug moves from the patch into the skin, and from the skin into the blood stream. The amount of drug per unit area remaining in the delivery device ( $A$ ) and the amount of drug per unit area in the skin ( $X$ ) are described by the pair of differential equations

$$(1) \quad \frac{dA}{dt} = -k_{21}A, \quad A(0) = A_0,$$

$$(2) \quad \frac{dX}{dt} = -k_{1B}X + k_{21}A, \quad X(0) = 0.$$

In these equations  $A_0$  is the initial concentration of drug per unit area in the delivery device. The parameter  $k_{21}$  is the rate at which drug enters the skin from the vehicle. The parameter  $k_{1B}$  is the rate at which drug leaves the skin and enters the blood stream.

- (a) Solve the system of differential equations to find the amount of drug per unit area in the vehicle and skin as a function of time. [7]  
 (Hint. First solve differential equation (1) and then substitute your answer into differential equation (2). Then solve differential equation (2). You may assume that  $k_{1B} \neq k_{21}$ ).
- (b) Find the value of time,  $t_{\max}$ , at which the concentration of drug in the skin ( $X$ ) reaches its maximum value. [3]
- (c) The cumulative amount of drug per unit area excreted from the skin into the blood-stream ( $Ae_s$ ) is expressed by

$$Ae_s(t) = k_{1B} \int_0^t X dt, \quad Ae_s(0) = 0.$$

Determine the cumulative amount of drug per unit area excreted from the skin into the blood stream in the limit that  $t \rightarrow \infty$ , i.e.

$$Ae_s(\infty) = k_{1B} \int_0^{\infty} X dt.$$

- (d) What is the significance of your answer to part (4c) of this question? [3]
- [1]

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5. The population of fish is modelled by the differential equation

$$\frac{dx}{dt} = f(x),$$

where the function  $f(x)$  is given by

$$f(x) = rx \left(1 - \frac{x}{K}\right) \left(\frac{x}{K_0} - 1\right),$$

where  $r > 0$  and  $0 < K_0 < K$ .

- (a) Sketch the growth curve  $f(x)$  as a function of  $x$ . [2]
- (b) Using your sketch determine the stability of the steady-state solutions  $x = 0$ ,  $x = K_0$  and  $x = K$ , carefully explaining your reasoning. [3]
- (c) How does the long-term evolution of the differential equation depend upon the choice of the initial condition  $x_0$ ? [3]
- (d) A disease spreads through the population reducing the population density to  $K_0/2$ . What happens to the population? Justify your answer. [2]

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6. The population density of the spruce budworm in a forest is given by the differential equation

$$\frac{dx}{dt} = rx \left( 1 - \frac{x}{q} \right) - \frac{x^2}{1+x^2}.$$

Determine the stability of the trivial steady-state solution  $x = 0$ . (Assume that  $r > 0$ ).  
[2]

7. Consider the logistic equation with constant effort harvesting

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) - Ex, \quad x(0) = K,$$

where  $x$  is the population density of an animal in an environment,  $t$  time,  $r > 0$  is the intrinsic growth rate,  $K > 0$  is the carrying capacity of the environment and  $E \geq 0$  is the effort expended in harvesting.

- (a) Determine the steady-state solutions of the logistic equation with constant effort harvesting. [3]
- (b) Suppose that the value for the intrinsic growth rate is  $r = 2$ .
  - (i) Determine the stability of the steady-state solutions as a function of the effort  $E$ . [3]
  - (ii) Draw a steady-state diagram for the logistic equation with constant effort harvesting for the case  $r = 2$  showing how the steady-state solutions of the model vary as a function of the effort expended in harvesting. Indicate stable and unstable steady-state solutions using solid and dashed lines respectively. [3]
  - (iii) For what value of the parameter  $E$  does a bifurcation occur? What kind of bifurcation is it? [2]

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8. Thermal and solutal dispersion in a circular tube with diffusion into the wall is characterised by a dispersion coefficient ( $\Lambda$ ) which is a function of the void fraction of the bed ( $\epsilon_f$ ) and the ratio of fluid thermal diffusivity to diffusivity in the wall ( $\mu$ ).

The function  $\Lambda$  is defined by the following maple code

```
L := g1/48.0 +mu*g2/8.0;  
g1 := epsilon*(6*epsilon^2 -16*epsilon+11);  
g2 := epsilon*(4*epsilon -epsilon^2 -3 -2*ln(epsilon));
```

where the function  $\Lambda$  is defined as L, the parameter  $\epsilon_f$  is defined as `epsilon` and the parameter  $\mu$  is defined as `mu`. In the following, you do *not* need to write out the definitions of the functions L, g1 and g2.

Write maple code to find the value of `epsilon` that maximises the function L, and the corresponding maximum value of the function L, when  $\mu = 0.1$ . [4]

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