University of Wollongong

School of Mathematics and Applied Statistics

MATH 111 — Applied Mathematical Modelling I (Wollongong Campus)

Spring Session Examination 2006

Family Name	______________
First Name	______________
Student Number	______________
Table Number	______________

Time Allowed: 2 hours and 15 minutes

Number of Questions: 8.

Directions to Candidates

1. Each question is to be attempted.
2. The questions are *not* of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on one side only.
4. WORKING (including all necessary reasoning) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed

Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied

None.

This examination paper must NOT be removed from the examination room.

1. Consider the difference equation

$$x_{n+2} - 3x_{n+1} + 2x_n = 0.$$

(a) State the order of this equation, whether it is linear or nonlinear and whether it is autonomous or non-autonomous. You must justify your answers. [1]

(b) Show that the general solution to the equation is of the form

$$x_n = A_1 + 2^n A_2,$$

where A_1 and A_2 are constants. [3]

(c) Using the general solution, solve for the specific solution with initial conditions $x_0 = 10$ and $x_1 = 20$. [2]

2. At the beginning of each year you invest $q\%$ of your yearly salary into a superannuation scheme. Your employer tops up your investment by adding $r\%$ of your salary. Your money draws interest of $p\%$ compounded yearly. Suppose that your salary starts at a base level b and at the start of each year it increases by a constant amount c. Your investment in the superannuation scheme after n payments is given by the solution of the difference equation

$$M_n = \left(1 + \frac{p}{100}\right) M_{n-1} + \left(\frac{q+r}{100}\right)(b + cn), \quad M_0 = \frac{q+r}{100} b.$$

(a) Find the general solution of the superannuation model, simplifying as far as possible. [5]

Hint. $\displaystyle\sum_{k=1}^{n} a^{n-k} k = \frac{a^{n+1} - (n+1)\,a + n}{(a-1)^2}$.

(b) Suppose that $q = 7$, $r = 14$, $b = \$50,000$, $c = \$1000$ and the rate of return from the superannuation scheme is 4%.

(i) What is the size of your initial investment? Round your answer to the nearest cent. [1]

(ii) You retire from work after making 35 payments. How much money do you have in your superannuation scheme? [2]

(c) How much money would you have in your superannuation scheme when you retired if you did not increase your contribution in line with your increasing salary, i.e. your regular investment is fixed at its initial value and $c = 0$. Assume that the values for $b, p, q \,\&\, r$ are as in part (b). [2]

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3. Explain in terms understandable to high-school students the concept of stability. Is this concept important? Justify your answer. [2]

4. In this question we consider a simple model for the one-off dumping of a pollutant (P) into a 'pond'. A side effect of dumping pollutant is that it removes oxygen from the pond. This scenario is represented by the model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k_p P, \quad P\,(t=0) = P_0, \tag{1}$$

$$\frac{\mathrm{d}O_2}{\mathrm{d}t} = -k_o P, \quad O_2\,(t=0) = 100\%. \tag{2}$$

In these equations k_p and k_0 are unspecified constants. In the absence of pollutant the oxygen in the pond is in equilibrium with atmospheric oxygen. The initial value of 100% represents the equilibrium value, in which the oxygen has been scaled so that it has no units.

(a) What does it mean in terms of the pond if there is a value of t such that $O_2\,(t) = 0$? [1]

(b) Solve the model.

Hint. First solve equation (1) to find $P\,(t)$. Then substitute this expression into equation (2). [7]

(c) Sketch a graph showing how the concentration of oxygen in the pond varies as a function of time. Mark on your graph the asymptotic value for the concentration as $t \to \infty$. [2]

(d) It is known that there is a threshold oxygen concentration in the pond, such that if the level of oxygen in the pond decreases below 30% then the fish in the pond will die. We must develop a decision-support procedure for the model.

(i) Show that there is a critical value of P_0, P_{cr}, such that if $P_0 \leq P_{\mathrm{cr}}$ then $\lim_{t\to\infty} O_2\,(t) \geq 30\%$. [3]

(ii) When $P_0 > P_{\mathrm{cr}}$ find the time, t_{cr}, it takes for the concentration of oxygen in the pond to reach 30%. [2]

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5. (a) The growth of a tumour inside the human body can be represented by the equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \beta T\left(1 - \frac{T}{K}\right), \quad T\left(0\right) = T_0.$$

where T is the size of the tumour, β denotes the growth rate of the tumour and K is the maximum tumour size.

(i) Sketch the rate of change of tumour growth $\dot{T}$ as a function of T. [2]

(ii) Using your sketch describe how the long-term evolution of the differential equation depends upon the choice of the initial condition T_0. [3]

(iii) Suggest biomedical interpretations for the steady-state solutions $T = 0$ and $T = K$. [1]

(b) The growth of a tumour inside the human body when radiation therapy is used can be represented by the equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \beta T\left(1 - \frac{T}{K}\right) - \mathcal{I}, \quad T(0) = T_0,$$

where the parameter $\mathcal{I}$ is proportional to the intensity of the radiation.

(i) Find the steady-state solutions of this model and sketch how they vary as a function of the intensity. (Do not calculate stability). [3]

(ii) Hence, or otherwise, determine a condition for the tumour to be destroyed. [1]

(iii) Suppose that for a particular patient $K = 1000$ and $\beta = 2$ (in appropriate units). Suppose that the value of the irradiance $\mathcal{I}$ can be controlled with an error tolerance of $\pm 1\%$. Suggest a value for $\mathcal{I}$ to destroy the tumour, justifying your answer. [2]

6. Determine graphically the stability of the trivial steady-state solution for the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax^3.$$

Consider the cases $a > 0$ and $a < 0$. What is the eigenvalue of the trivial steady-state solution? [4]

7. Consider the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x - 3x^2.$$

(a) Find the steady-state solutions of this equation and determine their stability as a function of the parameter μ. [5]

(b) Draw a steady-state diagram for this equation, indicating stable and unstable steady-state solutions using solid and dashed lines respectively. [2]

(c) For what value of the parameter μ does a bifurcation occur? [1]

8. In order to draw a cobwebbing diagram for the population model

$$x_{n+1} = \frac{2x_n^2}{x_n^2 + 3}$$

A figure containing the straight line $y = x$ and the curve $y = \dfrac{2x^2}{x^2+3}$ is required for a cobwebbing diagram.

(a) Why is the straight line $y = x$ required in a cobwebbing diagram? [1]

(b) Write appropriate maple code to plot the straight line $y = x$ and the curve $y = \dfrac{2x^2}{x^2+3}$ on the same figure over the range $0 \leq x \leq 10$. [2]

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