MATH 111 — Applied Mathematical Modelling I

Spring Session 2004

Mid-Session Test

Student Name: MARK NELSON *Student Number:* ____________

Instructions

Time Allowed: 90 minutes

- There are 9 questions of different value as marked. *All questions should be attempted.*
- *Working is to be done in the exam paper.*
- *At the end of the test the exam paper must be returned.*

Calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

This test paper is NOT to leave this room.

1. Consider the difference equation

$$y_{n+1} = f(y_n), \quad y_0 = 1, \quad n = 0, 1, \ldots$$

where $f(y_n)$ is some unspecified function of y_n. With reference to this equation explain what is meant by the word 'dynamics'. [1]

The word 'dynamics' means how the solution y_{n+1} varies with n.

2. (a) Give an example of an *autonomous* difference equation and a *non-autonomous* difference equation, explaining why your equation is autonomous/non-autonomous. [2]

$x_n = x_n + 1$ is an autonomous equation because the 'time' variable (here n) does not in the equation (except as a subscript). 1

$x_n = n x_n + 1$ is a non-autonomous equation because the 'time' variable (here n) appears in the equation in addition to being a supscript '$\underline{n} \cdot x_n$' 1

NO JUSTIFICATION = half marks

(b) Identify if the following difference equations are linear or non-linear. You *must* justify your answer. [2]

(i) $n_{y+2} = n_{y+1} y$ Linear. Each part of the equation is 'linear'. 1

(looks like 'n' or a constant)

(ii) $y_{n+1} = 2y_{n+1} + \sin(n)$ Linear. Each part of the equation is linear in 'y' – looks like 'n' or a constant. 1

NO JUSTIFICATION = NO MARKS

INCORRECT JUSTIFICATION = NO MARKS

This Ques should only be worth 1 mark.

3. Consider the problem of modelling the number of of chickens in Mr & Mrs Tweedy's farm. Each week the following activities occur:

- The number of chickens increases through natural growth by 10%.
- A fraction, α, of the chickens are killed by foxes.
- A constant number of chickens are converted into chicken pies.

(a) Write down a **word** equation that defines this problem. [2]

change in the number of chickens = growth in chickens − number killed by foxes − number converted into pies

(b) Write down, formally, the difference equation that describes the above scenario. Define **all** variables and explain your terms. [2]

C_{n+1} = number of chickens present in week $(n+1)$

C_n = number of chickens present in week(n)

P = number of chickens converted into pies

$$C_{n+1} - C_n = 0.1C_n - \alpha C_n - P$$

$$C_{n+1} = (1.1 - \alpha)C_n - P$$

1 mark for notation
1 mark for equation

4. How long will it take \$1500 to accumulate to at least \$2000 at 5.0% simple interest? [1]

$$S_n = \left(1 + \frac{np}{100}\right) S_0$$

$$\frac{S_n}{S_0} = 1 + \frac{np}{100}$$

$$\frac{100}{p}\left(\frac{S_n}{S_0} - 1\right) = n$$

$$n = \frac{100}{p}\left(\frac{S_n - S_0}{S_0}\right)$$

$$n = \frac{100}{5}\left(\frac{2000-1500}{1500}\right)$$

$$= 20\left(\frac{500}{1500}\right) = \frac{20}{3}$$

$n = \frac{20}{3}$ ~~months~~ years [or 7 years.]

$= 6\,^2/_3$ years

($^1/_2$ mark for formula if answer wrong)

5. Lien borrows \$20,000 to have a MATH111 chip implanted in her head so that everything makes sense. Interest is compounded monthly at 9% p.a.

   (a) In the first year Lien makes no repayments. How much does she owe at the end of the year? [1]

   (b) Starting in the second year Lien makes a repayment at the end of each month. If the loan is to be repaid after a further nine years what is the monthly repayment? [2]

a) Compound Interest Formula

$$S_{12} = S_0\left(1 + \frac{\alpha p}{100}\right)^n = 20{,}000\left(1 + \frac{9}{12 \times 100}\right)^{12}$$

$$= \$\,21876.13796$$

b) Mortgage Formula

$$D_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$ [½]

$D_{108} = 0$; $n = 108$; $\alpha = 1/12$; $p = 9$

$D_0 = S_{12}$ (from a) [½]

$$R = \frac{\frac{\alpha p}{100} \cdot \left(1 + \frac{\alpha p}{100}\right)^n D_0}{\left(1 + \frac{\alpha p}{100}\right)^n - 1}$$

$$= \$\,296.2665381$$ [1]

If answer to (a) is wrong can still get full marks for (b).

6. You have been given \$1000 to invest for one year. You have a choice of three bank accounts.

- 'You Beaut' bank offers you 10% compounded annually. At the end of the year you will pay \$20 in fees.
- 'Fair Go' bank offers you 10% compounded quarterly. At the end of the year you will pay \$30 in fees.
- 'Ocker' bank offers you 11% with payments made every four months. At the end of the year you will pay \$25 in fees.

Which bank should you put your money into (justify your answer)? [4]

Compound Interest formula

Money in bank = [Compound Interest Formula] − [Bank Fee]

<u>You Beaut</u>

$$S_1 = 1000\left(1 + \frac{1.10}{100}\right) - 20 = \$1080$$

<u>Fair Go</u>

$$S_4 = 1000\left(1 + \frac{10}{400}\right)^4 - 30 = \$1073.812891$$

<u>Ocker</u>

$$S_3 = 1000\left(1 + \frac{11}{300}\right)^3 - 25 = \$1089.082630$$

You get more bang for your buck at Ocker bank.

<u>No Justification = No Mark.</u>

[1 mark for correct bank if numbers are incorrect]

7. Patient flow in a department of geriatric medicine is modelled by the difference equation,

$$x_n = N + (1 - \alpha - \beta - \gamma)\, x_{n-1}, \quad n = 1, 2, 3 \ldots$$

where x_n is the number of patients in the department in the nth month, N is the number of new patients admitted each month, α is the fraction of current patients who are discharged each month, β is fraction of current patients who, unfortunately, die each month and γ is the fraction of the current patients who are transferred to another section each month. For convenience we write

$$a = 1 - \alpha - \beta - \gamma$$

and assume that $0 < a < 1$.

(a) Find the general solution of the patient flow model, simplifying as far as possible. [2]

The general sol^n to the equation

$x_n - a x_{n-1} = b(n)$ is $x_n = a^n x_0 + \sum_{p=1}^{n} a^{n-p} b(p)$.

Here $b(n) = N$ and the general solution is

$$x_n = a^n x_0 + \sum_{p=1}^{n} a^{n-p} \cdot N \quad [1]$$

$$= a^n x_0 + N \sum_{p=1}^{n} a^{n-p}$$

$$= a^n x_0 + N\left(\frac{a^n - 1}{a - 1}\right)$$ as $\sum_{p=1}^{n} a^{n-p}$ is a G.P. [1]

(b) What is the number of patients in the department in the limit $n \to \infty$? [2]

as $0 < a < 1$ then $\lim_{n\to\infty} a^n = 0$. [1]

Thus $$\lim_{n\to\infty} x_n = \frac{N \cdot (-1)}{a - 1}$$

$$= \frac{N}{1 - a} \quad [1]$$

(c) A new geriatric ward is added to a hospital. When the ward is opened there are no patients ($x_0 = 0$).

(i) Suppose that the new ward has 100 beds and it is anticipated that 50 patients are admitted a month.

- Explain why if $a \leq 0.5$ the ward never overfills. [2]

$$\lim_{n\to\infty} x_n = \frac{N}{1-a} \leq \frac{N}{1-0.5} \quad (\text{if } a \leq 0.5)$$

$$= \frac{N}{0.5}$$

$$= 2N$$

$$= 100 \quad \text{if } N = 50.$$

So, the number of occupied beds is not greater than the number of beds in the ward

- Give one reason why it is 'good' to operate with $a \leq 0.5$ and one reason why it is 'bad'. [2]

- good - running with spare capacity in case of emergencies
- bad - uneconomic to run with spare capacity

(ii) The ward operates with $a = 0.51$. During which month does the ward have to start turning patients away? [2]

$$x_n = N\left(\frac{a^n - 1}{a-1}\right) = 50\left(\frac{0.51^n - 1}{0.51 - 1}\right) = 100$$

$$\Rightarrow 0.51^n - 1 = 2(0.51-1)$$

$$0.51^n = 1 - 2 + 1.02 = 0.02$$

$$n \ln(0.51) = \ln(0.02)$$

$\Rightarrow n = 5.809.$ [1] Turn patients away in the sixth month [1]

(d) We have assumed that the parameters N & α are constant. Briefly discuss if this is reasonable. [2]

- N probably varies with the seasons - more ill people in winter.
- α probably o.k to have it fixed but it could change (e.g. a virus spreads the ward increasing the time it takes for patients to recover well enough to leave hospital)

(b) The stable fixed point of the harvesting model (should it exist) is given by

$$x^* = \frac{-(1-r) + \sqrt{(1-r)^2 - 4rh}}{2r}$$

Find the fixed point (to four decimal places), and the associated eigenvalue, when [3]

(i) $r = 1.5$ and $h = 0.015$.

(ii) $r = 1.6$ and $h = 0.05625$.

(iii) $r = 2$ and $h = 0.045$.

The eigenvalue of the fixed point x^* is given by

$\lambda = f'(x^*) = r(1-2x^*)$ ([1])

	x^*	λ
(i)	0.3	0.6
(ii)	0.1875	1.0
(iii)	0.45	0.2

[2]

[

$-1/2$ for saying (ii) is stable or unstable

~~the for~~

[1] for x^*

[1] for λ $1/2$ if $2/3$ λ are correct

8. Consider the logistic equation with fixed harvesting

$$x_{n+1} = rx_n(1 - x_n) - h, \quad n = 0, 1, 2 \ldots$$

where $1 < r < 4$ and $0 \leq h \leq 1$.

(a) Show that harvesting is only sustainable if

$$h \leq \frac{(r-1)^2}{4r}.$$

[4]

Find the fixed points.

Let $x_{n+1} = x_n = x^*$. [1]

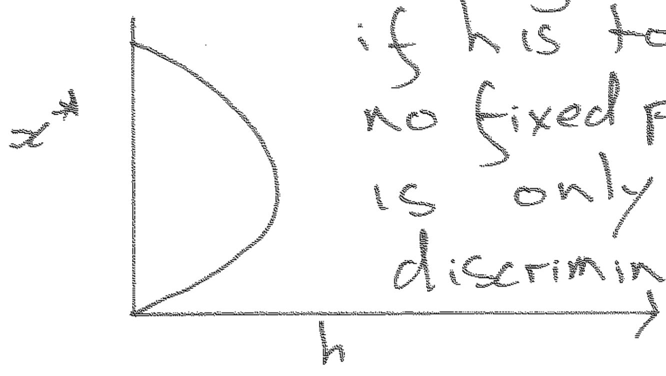
$x^* = rx^*(1-x^*) - h$ [1]

$x^* - rx^* + rx^{*2} + h = 0$

$rx^{*2} + (1-r)x^* + h = 0$

$$x^* = \frac{-(1-r) \pm \sqrt{(1-r)^2 - 4rh}}{2r} \quad [1] \quad (*)$$

Plotting x^* as a function of h, we see that



if h is too large there are no fixed points i.e harvesting is only sustainable if the discriminant of $(*)$ is non-negative.

$\therefore$ we require

$(1-r)^2 - 4rh \geqslant 0$

$$\Rightarrow h \leq \frac{(1-r)^2}{4r} \quad [1]$$

(c) Gollum Fresh Fish (motto 'fish fresh from the sea, three times a day') has the choice to send its fishing fleet to one of three fisheries. The value for h is regulated by Mordor Moguls. The long-term yearly profit ($\mathcal{P}$) for fishing in a fishery is

$$\mathcal{P} = ax^* - b$$

where a and b are parameters that depend upon the fishery and x^* is the steady fixed point of the harvesting model for the specified values of h and r. The numbers associated with each fishery are

Fishery one: $r = 1.5$, $h = 0.015$, $a = 3$, $b = 0.6$.
Fishery two: $r = 1.6$, $h = 0.05625$, $a = 4$, $b = 0.45$.
Fishery three: $r = 2$, $h = 0.045$, $a = 1$, $b = 0.2$.

Which fishery should Gollum Fresh Fish use and why? [2]

Calculate the profit at each fishery

[½ mark if 2/3 are correct]

(1) 0.3
(2) 0.3
(3) 0.25

$\}$ [1]

Gollum Fresh Fish should use fishery one because the eigenvalue is 0.6. At fishery two the eigenvalue is 1.0 - fishing is on the edge of stability. [1]

[Can still the 2nd mark if Q first ques wrong]

9. Find the fixed points of the Ricker difference equation. [2]

$$x_{n+1} = x_n \exp[r(1 - x_n)], \quad n = 0, 1, \ldots$$

Fixed are when $x_{n+1} = x_n = x$

i.e $x = xe^{r(1-x)}$ [1]

$x(1 - e^{r(1-x)}) = 0$

$\underline{\underline{x = 0}}$ or $1 - e^{r(1-x)} = 0$

[½]

$1 = e^{r(1-x)}$

$\ln 1 = \ln e^{r(1-x)}$

$0 = r(1-x) \Rightarrow \underline{\underline{x = 1}}$ [½]