MATH 111 — Applied Mathematical Modelling I

Spring Session 2003

Mid-Session Test

Student Name: MARK NELSON *Student Number:* ________

Instructions

Time Allowed: 90 minutes

- *All questions should be attempted.*
- *Working is to be done in the exam book.*
- *At the end of the test return the exam book.*

Calculators are permitted.

This test paper is NOT to leave this room.

1. Consider the difference equation

$$y_{n+1} = f(y_n), \quad y_0 = 1,$$

where $f(y_n)$ is some unspecified function of y. Explain what is meant by the word 'iteration'. [1]

Iteration is the process where we calculate the sequence

$$y_1 = f(y_0) = f(1)$$
$$y_2 = f(y_1) = f[f(y_0)]$$
$$y_3 = f(y_2) = f\{f[f(y_0)]\}$$

2.

(a) Define *autonomous* difference equation and *non-autonomous* difference equation. [2]

An autonomous difference equation is one in which the "time" subscript does not appear explicitedly in the equation.

A non-autonomous difference equation is one in which the "time" subscript appears explicitedly in the equation.

(b) Identify if the following difference equations are autonomous or non-autonomous. You *must* justify your answer. [3]

(i) $y_{n+2} = y_{n+1}y_n$

(ii) $y_{n+1} = 2y_{n+1} + \sin(n)$

(iii) $n_{k+2} = n_{k+1} - n_k$

i) Autonomous – n only appears as a subscript

ii) Non-Autonomous – n appears as sin(n)

iii) Autonomous – k only appears as a subscript

HALF ~~NO~~ MARKS for RIGHT answer without justification

3. For the difference equation

$$y_k = ky_{k-1}, k = 1, 2, 3 \ldots$$

and initial condition $y_0 = 1$.

(a) calculate y_1, y_2, y_3, y_4 and make a guess at the "closed-form" solution of y_k. [2]

$$\left.\begin{aligned} y_1 &= 1y_0 = 1 \\ y_2 &= 2y_1 = 2 \\ y_3 &= 3y_2 = 6 \\ y_4 &= 4y_3 = 24 \end{aligned}\right\} \quad 1$$

$$y_k = k! \qquad 1$$

(b) verify that your formula satisfies the difference equation and the initial condition. [2]

Initial Condition $y_0 = 0! = 1$ 1

$$\left.\begin{aligned} &y_k = k! \quad y_{k-1} = (k-1)! \\ &k! = k.(k-1)! \checkmark \end{aligned}\right\} \quad 1$$

The guess is wrong.

4. What is the simple interest earnt on \$5 000 invested for 48 months at 4.5%? [1]

$$S_4 = \left(1 + \frac{4 \times 4.5}{100}\right) S_0 = \left(1 + \frac{18}{100}\right) 5000 = 5900$$

Interest $= 5900 - 5000 = 900$

TRICK

5. Consider the problem of modelling patient flow in a department of geriatric medicine. Each day the following activities occur:

   - a number of new patients are admitted to the department for acute care.
   - A fraction of the current patients are treated and discharged.
   - A fraction of the current patients, unfortunately, die.
   - A fraction of the current patients are transferred to another section.

   (a) Write down a **word** equation that defines this problem. [2]

Change in patient number = new patients − patients discharged − patients died

− patients transferred

only current patients 01B10

+1 clever new patients = f(n)

1 for the idea
1 for all the details

   (b) Write down, formally, the difference equation that describes the above scenario. Define all variables and explain your terms. [2]

P_{d+1} patients at day $(d+1)$

P_d patients at day (d)

α - fraction of patients discharged

β - fraction of patients die

γ - fraction of patients discharged

N - new patients

1

$$P_{d+1} - P_d = N - \alpha P_d - \beta P_d - \gamma P_d$$

$$P_{d+1} = N + (1 - \alpha - \beta - \gamma) P_d \quad \text{— } 1$$

6. Steven decides to purchase a car for \$40 000. He has savings of \$17 000 and has the choice of two payment schemes.

   - He can put down a deposit of \$17 000 and take out a five-year loan (amortization scheme) from the bank with interest at 7.5% compounded quarterly.
   - He can put down a deposit of \$15 000 and make weekly payments of ~~\$80~~ 105 a year for five years to the dealer. At the end of five years he makes a final payment of ~~\$3500~~ ~~2000~~ 3500

   (a) Which option should Steven choose (justify your answer)? How much money does he save? [5]

Scheme 1
Find repayments

$$0 = (1+i)^n\left(D_0 - \frac{R}{i}\right) + \frac{R}{i}$$

$$R\left\{\frac{(1+i)^n}{i} - \frac{1}{i}\right\} = (1+i)^n D_0$$

$$R = \frac{i(1+i)^n D_0}{(1+i)^n - 1} \qquad i = \frac{\alpha p}{100}$$

$\alpha = \frac{1}{4}$; $p = 7.5$; $D_0 = 23000$

$n = 20$

1/2 - wrong numbers

$R = 1309.69$ $\boxed{1}$

Total paid = $17000 + 20 \times 1309.69$

$= 44793.80$ $\boxed{1}$

Go with the bank - it's cheaper $\boxed{1}$

1/2 if wrong, but consistent

Save \$1006.12 $\boxed{1}$

Scheme 2 ②

$15000 + \cancel{80} \; 105 \times 52 \times 5$

$+ 3500$ $\quad 45,800$ $\boxed{1}$

LOAN REPAYMENTS not compound interest

(b) Steven opts to pay the dealer directly rather than take a loan out from the bank. He decides to invest the remaining \$2 000 of his savings in a five-year term deposit account with his bank. If interest is compounded annually what is the minimum interest rate that is required for his decision to make sense? [2]

Needs to accumulate 1006.12 in interest over 5 years 1

$$(2000 + 1006.12) = \left(1 + \frac{p}{100}\right)^5 (2000)$$

$$\Rightarrow p = 8.49\%$$ 1

7. Find the solution of the following difference equation, simplifying as far as possible. Carefully explain each step of your solution. [4]

$$x_n - x_{n-1} = n^2, \quad x_1 = 2$$

Use the general formula with $a = 1$ and $b(n) = n^2$ 1

$$x_n = x_0 + \sum_{k=1}^{n} p^2$$ 1

$$x_n = x_0 + \frac{n(n+1)(2n+1)}{6}$$

Now $x_1 - x_0 = 1$

$$\to 2 - 1 = x_0$$

$$x_0 = 1$$ 1

$$x_n = 1 + \frac{n(n+1)(2n+1)}{6}$$ 1

8. Consider the population model

$$x_{t+1} = f(x_n).$$

(a) What does it mean for the point x^* to be a fixed point of this equation? [1]

$x^* = f(x^*)$

(b) Why is a fixed-point called a fixed-point? [1]

Because if $x_n = x^*$ then

$x_{n+1} = f(x_n) = x_n$

(c) Explain why it is important to find the fixed points of this equation if we want to study the long-term dynamics of this model [1]

Fixed points represent steady-state (static) solutions of the model.

9. In this question we consider the discrete logistic equation with fixed harvesting

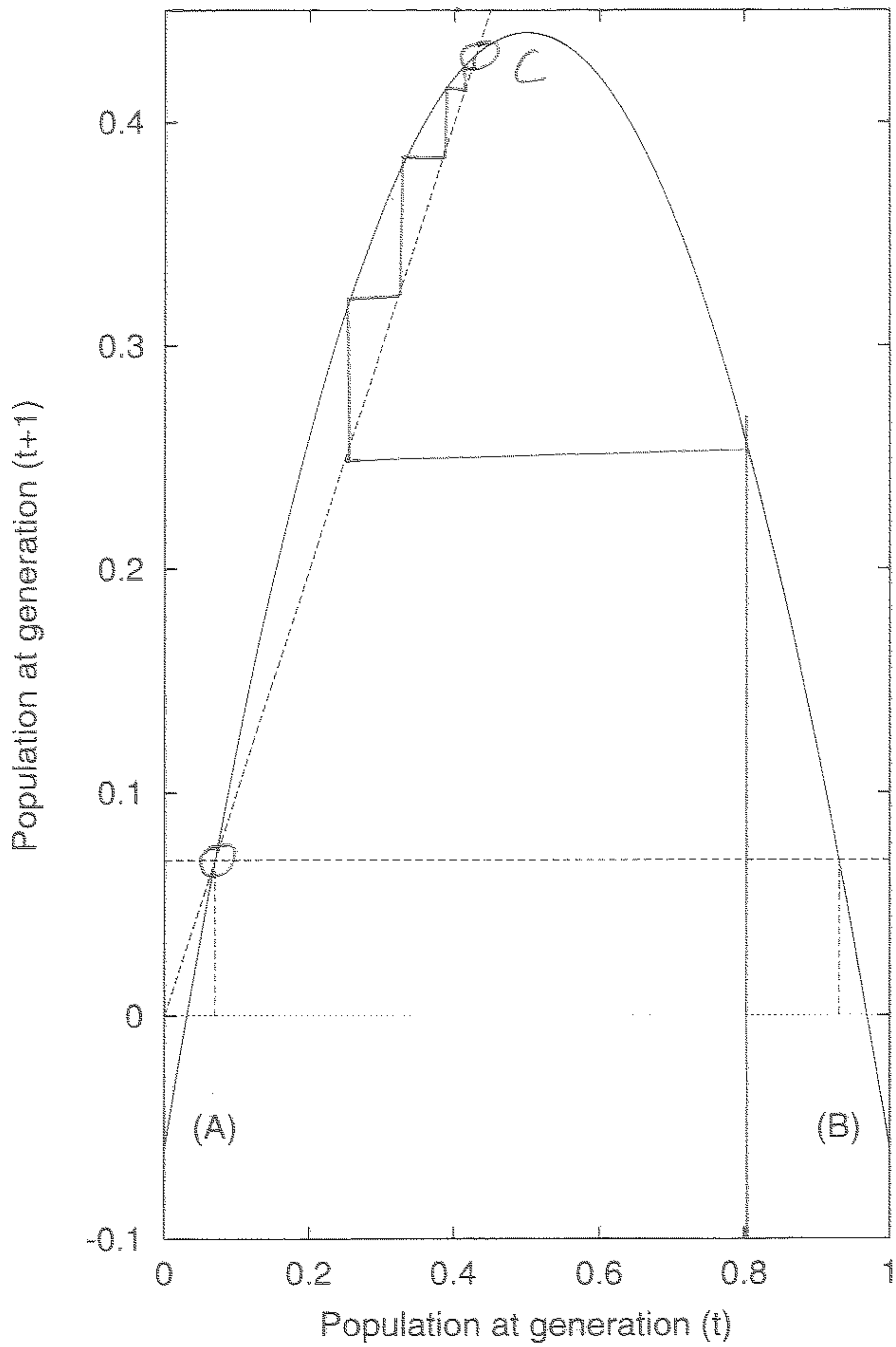
$$x_{n+1} = rx_n\,(1 - x_n) - h$$

with $r = 2$ and $h = 0.06$.

(a) Identify the location of the fixed point(s) of this map on the diagram below. [1]

[illegible]

O - fixed points



(b) By drawing successive iterations on the cobweb diagram above determine the long-term evolution of the point $x_0 = 0.8$ [1]

It goes to the fixed point marked C

(c) Explain what your cobweb means biologically. with regard to harvesting [1]

As $n \to \infty$ the population approaches the value C.

"it doesn't look good they will die out"

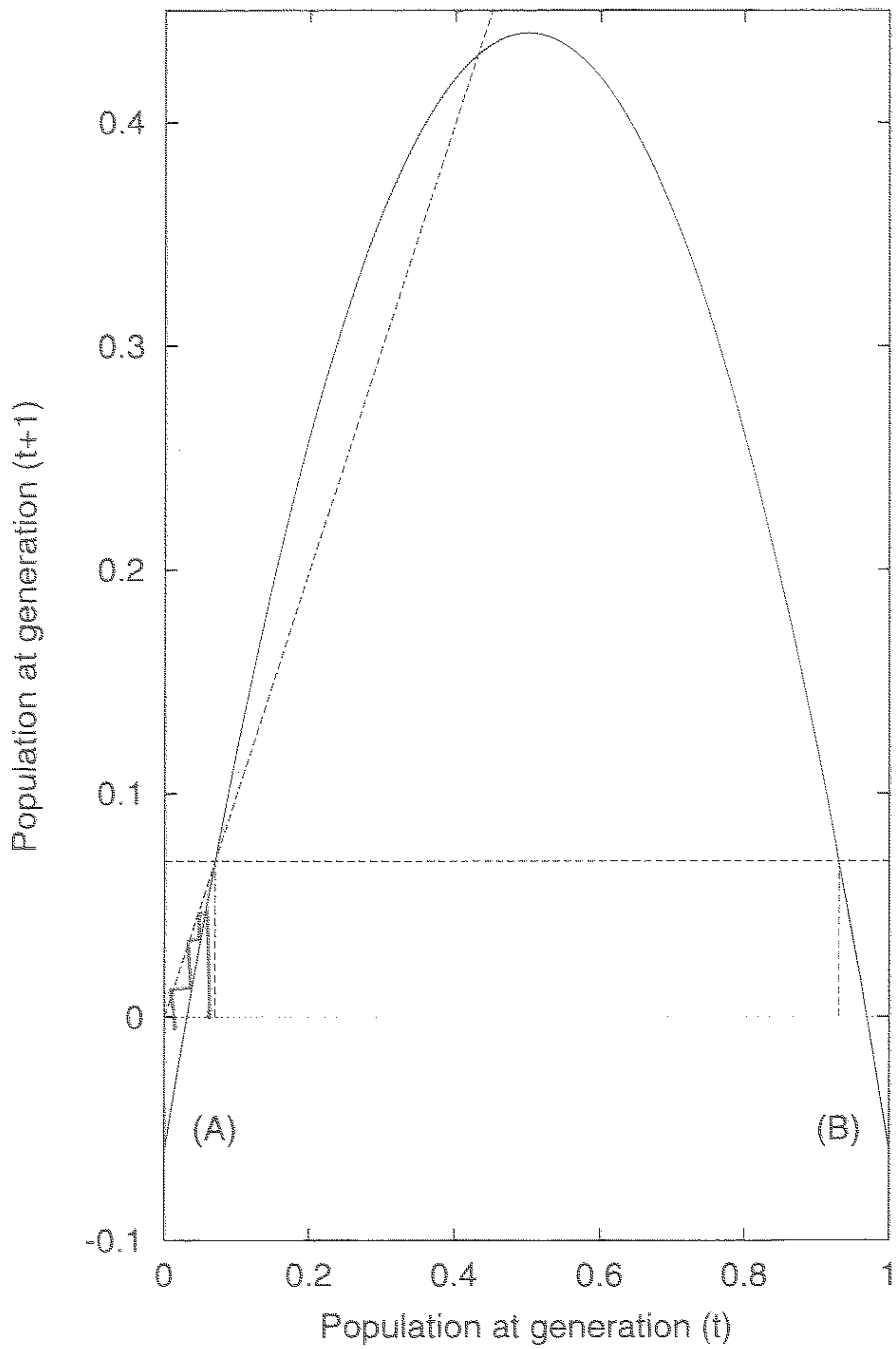
(d) How would your answer to (c) change if you were choose a different value for x_0 with x_0 between the values $x_0 = A$ and $x_0 = B$? ($A = 0.0697$ and $B = 0.9303$, marked on the figure). [1]

The long-term behaviour would be the same.

$\boxed{x_0 \in (A, B)}$

$\rceil$ not $\lfloor$

(e) By drawing successive iterations on the cobweb diagram below determine the long-term evolution of the point $x_0 = B$, where B is any point in the range $0 < x_0 < A$, where $A = 0.0697$ is marked on the figure. [1]



(f) Explain what your cobweb means biologically. [1]

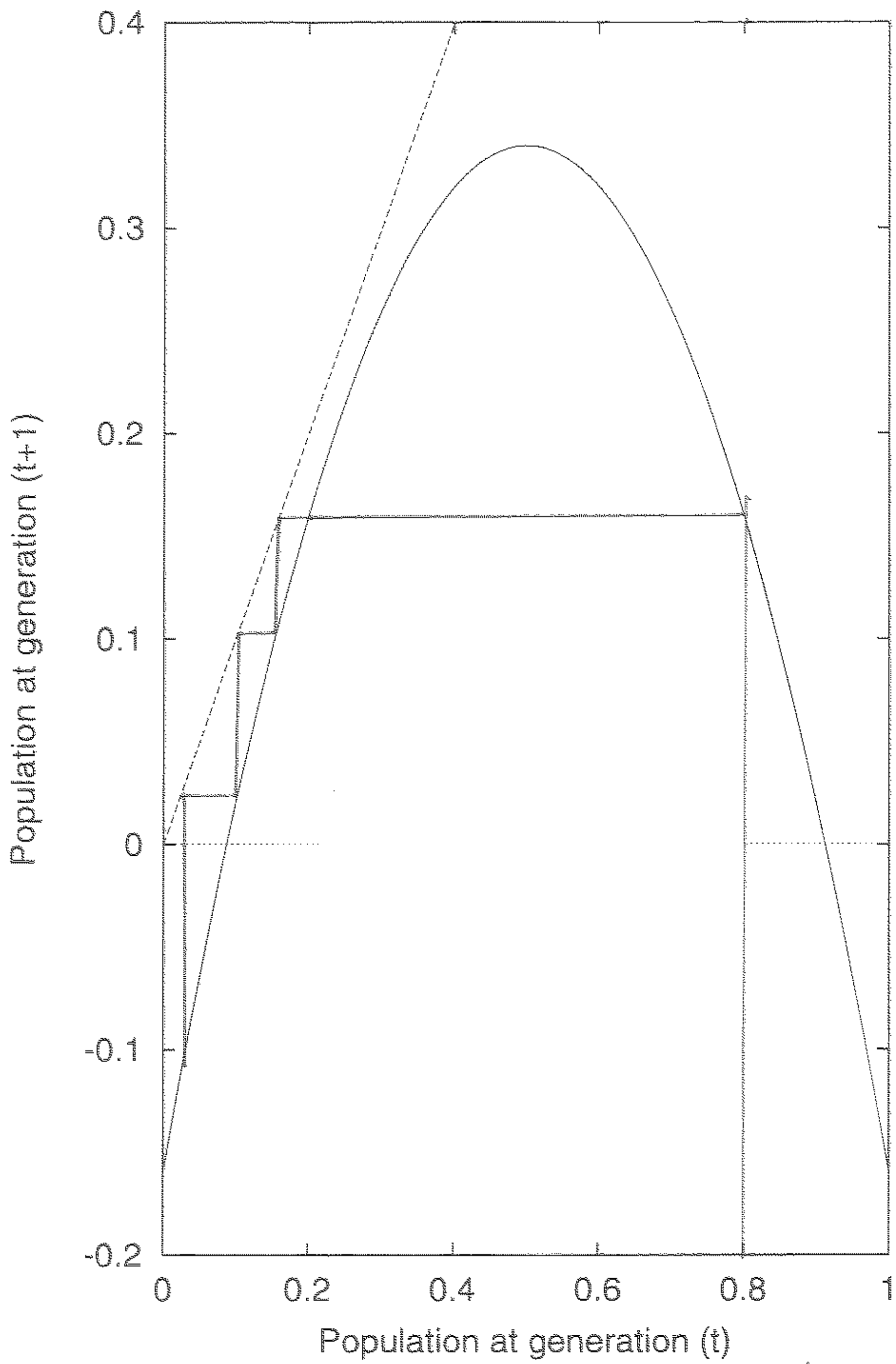
Population becomes extinct.

10. In this question we consider the discrete logistic equation with fixed harvesting

$$x_{n+1} = rx_n\,(1 - x_n) - h$$

with $r = 2$ and $h = 0.16$.

(a) By drawing successive iterations on the cobweb diagram below determine the long-term evolution of the point $x_0 = 0.5$, [1]



(b) Explain what your cobweb means biologically. Population becomes extinct. That will all the [1]

(c) How would your answer to (b) change if you were choose a different value for x_0 with $x_0 > 0$. [1]

It wouldn't. Population always becomes extinct.

everything has died.

perceptive

+ Bonus → doesn't make sense $\frac{0\text{-}08?}{70\text{-}91?}$.

11. Find the fixed points of the logistic difference equation. [2]

$$x_{n+1} = rx_n\,(1 - x_n)$$

$x^* = f(x^*)$

$x = rx(1-x)$ $\qquad$ $x = 0$!!!

$x - rx(1-x) = 0$

$x\,[1 - r(1-x)] = 0$

$x = 0$ or $1 - r(1-x) = 0$

$\frac{1}{r} = 1 - x$

$x = 1 - \frac{1}{r}$

$= \frac{r-1}{r}$

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A Formulae

A.1 First order difference equations

1. The first-order difference equation
$$x_n - a x_{n-1} = b(n)$$
has solution
$$x_n = x_0 a^n + \sum_{p=1}^{n} a^{n-p} b(p)$$

2. (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

   (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   (c) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

   (d) $\displaystyle\sum_{k=1}^{n} k^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

3. the sum of a geometric progression
$$k, rk, r^2k, r^3k \ldots .$$
is
$$S_n = \frac{k(r^n - 1)}{r-1} \quad \text{where } r \neq 1$$

A.2 Financial mathematics

In the following formulae p is the interest rate and α is the fraction of the year occupied by a conversion period.

The simple interest formula is
$$S_n = \left(1 + \frac{np}{100}\right) S_0$$
where S_n is the amount of money in the bank after n years S_0 is the amount invested.

The compound interest formula is
$$S_n = \left(1 + \frac{\alpha p}{100}\right)^n S_0$$
where S_n is the amount of money in the bank after n payments and S_0 is the amount invested.

The loan repayment (amortization scheme) formula is
$$D_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(D_0 - \frac{100R}{\alpha p}\right) + \frac{100R}{\alpha p}$$
where D_n is the debt after n payments, n is the number of conversion periods, D_0 is the amount borrowed and R is the repayment made at the each of each conversion period.

The annuity formula is
$$y_n = \left(1 + \frac{\alpha p}{100}\right)^n \left(y_0 + \frac{100R}{\alpha p}\right) - \frac{100R}{\alpha p}$$
where y_n is the the amount of money in the annuity after n payments, n is the number of conversion periods, y_0 is the initial amount invested in the annuity and R is the payment made at the each of each conversion period.