) A differential equation is linear if 07-10-0[illegible]
every derivative is linear in the dependendent verrable and if any term not including a derivative is linear in the dependent verrable.

A differential equation is non-linear if ~~the~~ any term involving a derivative or the ~~dependent~~ dependent variable is non-linear in the dependent verrable

b) Linear - each term is linear in the dependent variable (y)

Non-linear - The term $\frac{1}{y}$ is non-linear in the dependent variable

Linear - each term is linear in the dependent verrable variable (t)

7 min

~~If [illegible] of no justif~~

~~Only [illegible]~~ if no justification

0 marks - no justif. /

$$S_n = \left(1 + \frac{\alpha p}{100}\right)^n S_0$$

$\alpha = 1/4$
$p = 9$
$n = 8$

$$= \left(1 + \frac{\frac{1}{4} \times 9}{100}\right)^8 10{,}000$$

$$= \cancel{11{,}428.25} \quad 11940.31$$ 1

Interest $=$ $11{,}428.25 - 10{,}000.00$

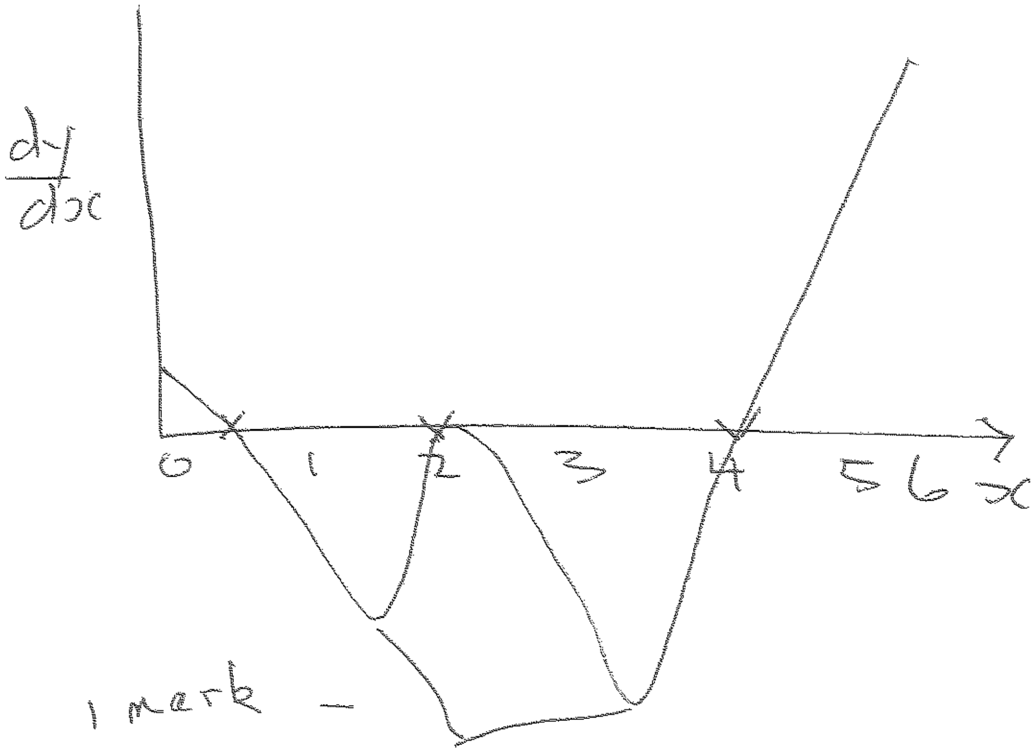
$\cancel{1428}.25$ 1

1940.3

3 a) x^* is a steady-state if $f(x^*) = 0$. 1

b) steady-state solutions represent ~~static~~ static solutions to the model 1

4.



1 mark –

$\frac{d^2y}{dx^2} = 0$

1 mark – regions where $\frac{dy}{dx} > 0$

1 mark – regions where $\frac{dy}{dx} < 0$.

2) a] The population is decaying towards zero
The population will become extinct.
The population decays exponentially to zero
1.5

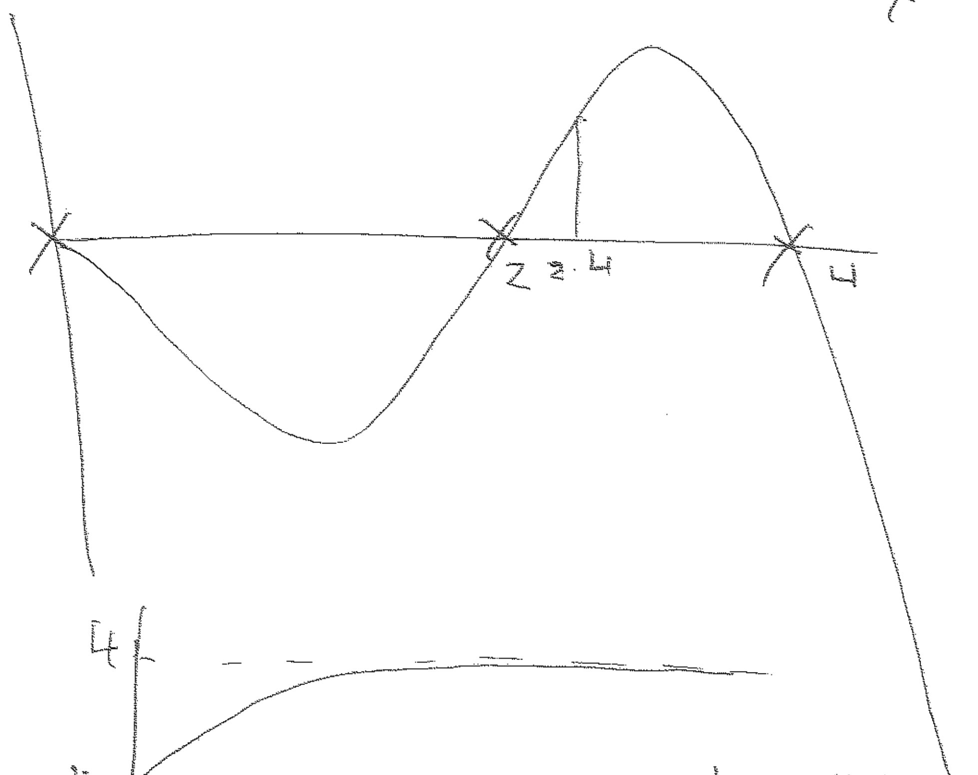
b] The population shows damped oscillations to a fixed point.
The population survives at a non-zero value.
Yes, from the figure it appears that
$x_n \to 0.62$ as $n \to \infty$.
1.5

c] The population converges to a period-2 solution.
One year the pop is 0.55, the next year it is 0.75.
Yes, a period 2 solution
1.5

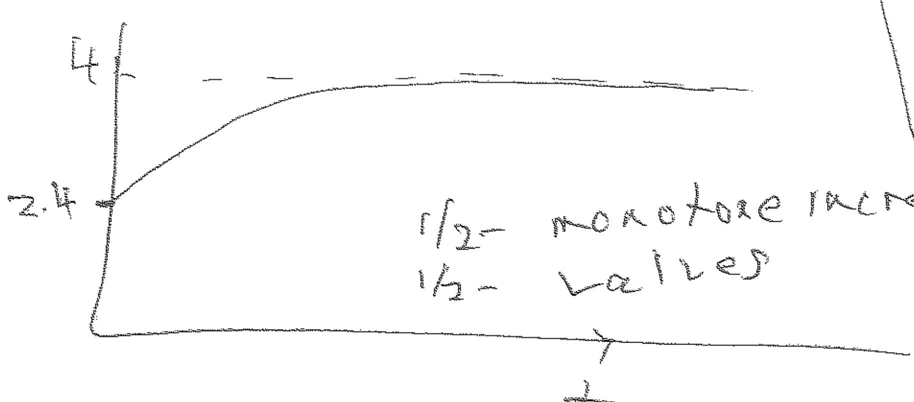
d] The solution appears chaotic.
The population varies in a random manner.
No - it's impossible to make a detailed prediction.

59

x = steady-state



b)



1/2 - monotone increase
1/2 - values

c) Population tends towards the steady-state value $x^* = 4$.

d) If $v \in [0, 2)$ (1)

then $v \to 0$ as $t \to \infty$ (1)

i.e the population becomes extinct.

If $v \in [2, 5)$ (1) $v \to 4$ as $t \to \infty$ (1)

if $v_0 = 2$ then $v = 2 \ \forall t$ (1) if $v_0 = 4$ $v = 4 \ \forall t$ (1)

BONUS MARK for saying

$v \in (2, 4)$ v increases monotonically.

BONUS MARK for saying $v \in (4, 5)$

v decreases monotonically

e) we know that if $v_0 \in (0,2)$ then $v \to 0$ as $t \to \infty$. $\therefore$ harvesting strategy is to reduce the population to a value v with $0 < v < 2$.

7.)

rate of change of pop^n = rate of births − rate of deaths

) $\frac{dX}{dt} = \beta X - \alpha X$; $X(t=0) = X_0$

BONUS MARKS

1/2

) $\frac{dX}{dt} = (\beta - \alpha)X$ + BONUS

$\frac{dX}{X} = (\beta - \alpha)dt$

$\ln\left[\frac{X}{X_0}\right] = (\beta - \alpha)t$

$X = X_0 \exp[(\beta - \alpha)t]$ 1.

d) if $\beta > \alpha$ then $X \to \infty$ as $t \to \infty$

if $\beta = \alpha$ then $X = X_0 \ \forall t$

if $\beta < \alpha$ then $X \to 0$ as $t \to \infty$.

8.)

$$\frac{dI}{dt} = 0 \Rightarrow \beta I\left(1 - \frac{I}{K}\right) - \alpha I = 0$$

$$I\left\{\beta\left(1 - \frac{I}{K}\right) - \alpha\right\} = 0$$

$I = 0$ or $\beta\left(1 - \frac{I}{K}\right) - \alpha = 0$

1

$$1 - \frac{I}{K} = \frac{\alpha}{\beta}$$

$$1 - \frac{\alpha}{\beta} = \frac{I}{K}$$

$$I = K\left(1 - \frac{\alpha}{\beta}\right)$$ ½

stability given by eigenvalue λ

where $\lambda = \frac{df}{dI}(I^*)$ 1

$$f = \beta I\left(1 - \frac{I}{K}\right) - \alpha I$$

$$\frac{df}{dI} = \beta\left(1 - \frac{2I}{K}\right) - \alpha$$

when $I = 0$

$\lambda = \beta - \alpha$ STABLE if $\beta - \alpha < 0$

$\boxed{\beta < \alpha}$

1

when $I = K\left(1 - \frac{\alpha}{\beta}\right)$

$$\lambda = \beta\left(1 - \frac{2K}{K}\left(1 - \frac{\alpha}{\beta}\right)\right) - \alpha$$

$$= \beta - 2\beta\left(1 - \frac{\alpha}{\beta}\right) - \alpha$$

$$= \beta - 2\beta + 2\alpha - \alpha$$

$= \alpha - \beta$ STABLE if ~~$\beta > \alpha$~~ $\alpha - \beta < 0$

b) disease dies out if only s.state is $I = 0$, i.e $\alpha - \beta < 0$

a) $C = 0.1 c_0$

$0.1 c_0 = c_0 \exp\left(\frac{-q}{V} t\right)$ ①

$0.1 = \exp\left(\frac{-q}{V} t\right)$

$\ln(0.1) = -\frac{q}{V} t$

$t = \frac{-V}{q} \ln(0.1)$ ②

$\frac{4}{27} = \frac{27}{x}$

$x = \left(\frac{27^2}{4}\right)$

$\doteq 2.3 V/q$ ①

1b) $t = \cancel{\frac{-(27\times10^6)}{(4\times10^6)} \ln(0.1)} = \frac{27\times10^6}{4\times10^6} \ln(10)$

$= -\frac{(1.401\times10^6)}{10^7} \ln(0.1) = 0.341$ months.

$\frac{1.401}{10}$

c) 28 days in a month

$28 \times 0.341 = 9.540$ ✓

$30 \times 0.341 = 10.230$ ✗

BONUS mark - 29 days = 9.889 = OK.

10)

$$x_n = N + a x_{n-1} \qquad x_n - a x_{n-1} = N$$

$$x_n = x_0 a^n + \sum_{p=1}^{n} a^{n-p} b(p)$$

$$a = a \qquad b(p) = N$$

$$x_n = x_0 a^n + N \sum_{p=1}^{n} a^{n-p} \qquad 1$$

$$= x_0 a^n + \frac{N(a^n - 1)}{a - 1} \qquad 1$$

b) $a < 1$ so as $n \to \infty$ $a^n \to 0$ 1

$$\lim_{n\to\infty} x_n = \frac{-N}{a-1} = \frac{N}{1-a} \qquad 1$$

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c). $\lim_{n\to\infty} x_n = \frac{N}{1-a} = \frac{50}{1-a}$ 1

if $a < 0.5$ then $\frac{50}{1-a} < 100$ 1

d) $x_n = \frac{N(a^n - 1)}{a-1}$

Yes or No
+1 Justification
but no 1 for Y/N
if no justification.

$$\left(\frac{x_n}{N}\right)(a-1) = a^n - 1$$

$$1 + \left(\frac{x_n}{N}\right)^{(a-1)} = a^n$$

$$\ln\left[1 + \left(\frac{x_n}{N}\right)^{(a-1)}\right] = N$$

$$n = \frac{\ln\left[1 + 2(0.51 - 1)\right]}{\ln[0.51]} = \frac{\ln[0.02]}{\ln[0.51]}$$

$n = 5.8$

In the 6th month (1)

a) N – ~~total~~ number of new patients. Would expect to be greater in winter, less in summer.

α – maybe constant is ok, but perhaps winter = more.

β – constant is a good assumption

γ – constant is a good assumption

Any reasonable answer = marks.

Steady states

$$x = rx(1-x) - h \qquad 1$$

$$x = rx - rx^2 - h$$

$$rx^2 + (1-r)x + h = 0$$

$$x = \frac{-(1-r) \pm \sqrt{(1-r)^2 - 4rh}}{2r} \qquad 1$$

Not sustainable if $(1-r)^2 - 4rh < 0$ 1

i.e $(1-r)^2 < 4rh$

$$h > \frac{(1-r)^2}{4r}.$$

b) i) $x = \dfrac{-(1-1.5) + \sqrt{(1-1.5)^2 - 4(1.5)(0.015)}}{2(1.5)}$

$= 0.3$ 0.12

ii) $x = \dfrac{-(1-1.6) + \sqrt{(1-1.6)^2 - 4(1.6)(\cancel{0.0075}\ 0.05625)}}{2(1.6)}$

0.1875

iii) $x = \dfrac{-(1-2) \pm \sqrt{(1-2)^2 - 4(2)(0.045)}}{4}$

$x = 0.45$

1.) $3 \times 0.3 - 0.6 = 0.30$ $r = 1.5$

2.) $4 \times 0.1875 - 0.45 = 0.30$ $r = 1.6$

3.) $1 \times 0.45 - 0.2 = 0.25$ $r = 2$

Fishery 1 - Fishery 2 is on the edge of being unsustainable and any disaster could lead to the stock being wiped out.

$\boxed{\text{Bonus!}}$