Math302: Exam style question

Q1

a) $$u'' + u + \epsilon (u')^3 + \epsilon u^3 = 0$$

$$u = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1)$$

$$u'' = u_{T_0T_0} + 2\epsilon u_{T_0T_1}$$

$$= u_{0T_0T_0} + \epsilon u_{1T_0T_0} + 2\epsilon u_{0T_0T_1}$$

$$u' = u_{0T_0}$$

$$\Rightarrow \quad u_{0T_0T_0} + \epsilon u_{1T_0T_0} + 2\epsilon u_{0T_0T_1}$$
$$+ u_0 + \epsilon u_1 + \epsilon (u_{0T_0})^3 + \epsilon (u_0)^3 = 0$$

$O(1)$ $$u_{0T_0T_0} + u_0 = 0$$

$O(\epsilon)$ $$u_{1T_0T_0} + u_1 = -2u_{0T_0T_1} - u_0^3$$
$$- u_{0T_0}^3$$

$$u_0 = a(T_1)\cos(T_0 + b(T_1))$$
$$u_{0T_0} = -a\sin(T_0 + b)$$

$$u_{1T_0T_0} + u_1 = -a^3\cos^3(T_0+b)$$
$$+ a^3 \sin^3(T_0+b)$$

$$+2a_{T_1} \sin(T_0+b)$$
$$+ 2ab_{T_1} \cos(T_0+b)$$

$$= -a^3 \tfrac{3}{4} \cos(T_0+b) - a^3 \tfrac{1}{4} \cos(3T_0+3b)$$
$$+ \tfrac{3}{4} a^3 \sin(T_0+b) - a^3 \frac{1}{4} \sin(3T_0+3b)$$
$$+ 2a_{T_1} \sin(T_0+b)$$
$$+ 2ab_{T_1} \cos(T_0+b)$$

match secular terms:

$\sin(T_0+b)$: $\quad \tfrac{3}{4} a^3 = -2a_{T_1}$

$\cos(T_0+b)$: $\quad \tfrac{3}{4} a^3 = 2ab_{T_1}$

$$\frac{da}{dT_1} = -\frac{3}{8}a^3$$

$$\frac{1}{a^3}da = -\frac{3}{8}dT_1$$

$$\Rightarrow \quad a^{-2} = \frac{3}{4}T_1 + c$$

$$a = \frac{1}{\sqrt{\frac{3}{4}T_1 + c}}$$

$$\frac{db}{dT_1} = \frac{3}{8}a^2 = \frac{3}{8}\left(\frac{1}{\frac{3}{4}T_1 + c}\right)$$

$$\Rightarrow \quad b = \frac{1}{2}\ln\left(\frac{3}{4}T_1 + c\right) + d$$

$$\therefore \; u = \frac{1}{\sqrt{\frac{3}{4}T_1 + c}}\cos\left(T_0 + \frac{1}{2}\ln\left(\frac{3}{4}T_1 + c\right) + d\right)$$

b)

$$\theta = t + \frac{1}{2} \ln\left(\tfrac{3}{4}\epsilon t + c\right) + d$$

expand in a Taylor series

$$\theta = t + \frac{1}{2}\ln(c) + d + \frac{3}{8c}\epsilon t \; \ldots$$

$$\therefore \quad \omega = 1 + \frac{3\epsilon}{8c}$$

c) 5%.

at $t=0$ $\quad a = \left. \dfrac{1}{\sqrt{\tfrac{3}{4}\epsilon t + c}} \right|_{t=0}$

$$= \frac{1}{\sqrt{c}}$$

5% is $\dfrac{1}{20\sqrt{c}}$

$$\Rightarrow \quad \frac{1}{20\sqrt{c}} = \frac{1}{\sqrt{\tfrac{3}{4}\epsilon t + c}}$$

$$\Rightarrow \quad t = \tfrac{4}{3}\frac{(399c)}{\epsilon}$$